

Rock blocks.

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Dedicated to the memory of Joe Silk.

Abstract

Consider representation theory associated to symmetric groups, or to Hecke algebras in type A, or to q -Schur algebras, or to finite general linear groups in non-describing characteristic. Rock blocks are certain combinatorially defined blocks appearing in such a representation theory, first observed by R. Rouquier. Rock blocks are much more symmetric than general blocks, and every block is derived equivalent to a Rock block. Motivated by a theorem of J. Chuang and R. Kessar in the case of symmetric group blocks of abelian defect, we pursue a structure theorem for these blocks.

Contents.

Introduction.

- I. Highest weight categories, q -Schur algebras, Hecke algebras, and finite general linear groups.
- II. Blocks of q -Schur algebras, Hecke algebras, and finite general linear groups.
- III. Rock blocks of finite general linear groups and Hecke algebras, when $w < l$.
- IV. Rock blocks of symmetric groups, and the Brauer morphism.
- V. Schur-Weyl duality inside Rock blocks of symmetric groups.
- VI. Ringel duality inside Rock blocks of symmetric groups.
- VII. James adjustment algebras for Rock blocks of symmetric groups.
- VIII. Doubles, Schur super-bialgebras, and Rock blocks of Hecke algebras.
- IX. Power sums.
- X. Schiver doubles of type A_∞ .

References.

Introduction

Whilst the 20th century was still in its infancy, an article by F. G. Frobenius was published in the Journal of the Berlin Science Academy, which contained a description of the irreducible complex characters of all symmetric groups [34]. Since then, representation theory has evolved into a deep and sophisticated art, to the point where most papers in the subject are incomprehensible to the multitude of mathematicians. After all this development however, some basic questions remain unanswered. Interrogate an expert on group representation theory over finite fields and you will quite soon witness a shrug of the shoulders, and a protestation of ignorance. The irreducible characters of symmetric groups, which Frobenius so casually exposed in characteristic zero, remain mysterious over fields of prime characteristic. This monograph comprises a sequence of reflections surrounding the modular representation theory of symmetric groups.

Our approach to the subject is homological, inspired by M. Broué's abelian defect group conjecture [9], and encouraged by the proof of Broué's conjecture for blocks of symmetric groups by J. Chuang, R. Kessar, and R. Rouquier.

The abelian defect group conjecture is the most homological of a menagerie of general conjectures in modular representation theory, each of which predicts a likeness between the representations of a finite group in characteristic l , and those of its l -local subgroups. It stakes that the derived category of any block A of a finite group is equivalent to the derived category of its Brauer correspondent B , so long as the blocks have abelian defect groups. Such an equivalence should respect the triangulated structure of the derived category, and therefore descend from a two sided tilting complex of A - B -bimodules, by a theorem of J. Rickard [62]. Some have postured to prove the conjecture by induction, and encountered the difficulty of lifting an equivalence of stable categories to an equivalence of derived categories [65]. Others have tried to prove the conjecture for particular examples, such as symmetric groups.

In 1991, R. Rouquier observed a certain class of blocks of symmetric groups, which he believed to possess a particularly simple structure. Indeed, Rouquier conjectured a beautiful structure theorem for such blocks of abelian defect, which was subsequently proved by J. Chuang, and R. Kessar [11]. A corollary was a proof of Broué's conjecture for this class of blocks, whose defect groups could be arbitrarily large. In view of their history, these blocks should properly be called Rouquier, or Chuang-Kessar blocks. We

use the curt abbreviation “RoCK blocks”. Such blocks can be defined in arbitrary defect, and in any species of type A representation theory.

Chuang and Rouquier proved in a later work that all symmetric group blocks of identical defect possess equivalent derived categories which, in conjunction with the previous study of Rock blocks, established the truth of Broué’s conjecture for all blocks of symmetric groups [12].

The will which motivated this text, was for a theorem like that of Chuang and Kessar, describing Rock blocks whose defect groups are not necessarily abelian. Such a result ought to be of broad interest since there is no known analogue of Broué’s conjecture in nonabelian defect.

So far as symmetric groups are concerned, an algorithm of A. Lascoux, B. Leclerc and J-Y. Thibon gives a conjectural description of all the decomposition numbers of blocks of abelian defect [52]. This algorithm was formulated upon the examination of numerous tables of decomposition numbers for small symmetric groups, made by G. James [46]. The algorithm has been proven to describe the decomposition numbers for Hecke algebras at a root of unity in characteristic zero, by S. Ariki [3]. In nonabelian defect however, little is currently known about the decomposition numbers, even conjecturally. Symmetric group blocks of nonabelian defect are therefore beyond the influence of the general character theoretic predictions for algebraic groups made by G. Lusztig.

Over the length of this monograph, I hope to convince the Reader of the existence of a structure theorem for Rock blocks of arbitrary defect. Indeed, a conjectural description of an arbitrary Rock block of a Hecke algebra appears in chapter 8 (conjecture 165). A generalized conjecture for blocks of q -Schur algebras is given in chapter 9 (conjecture 178).

In formulating these conjectures, I produced more than a cupful of imaginative sweat, and earlier chapters of this booklet record theorems which point to the conjectures, and give evidence for them. For those readers with a fetish for decomposition numbers, the saltiest of these theorems is probably a formula for the decomposition matrix of a Rock block of a symmetric group, of arbitrary defect (theorem 132).

It would be polite of me to be a little more precise. Therefore, let us catalogue the more significant results of the article, and give some description of their character and logical intimacy, before plunging into the depths of the text.

In the first chapter, we recall E. Cline, B. Parshall, and L. Scott’s definition of a highest weight category. We discuss quasi-hereditary algebras,

and summarise C. Ringel's tilting theory for these algebras. We recognise the q -Schur algebra $\mathcal{S}_q(n)$ as the graded dual of the quantized coordinate ring of a matrix algebra, and recall S. Donkin's tilting theory for $\mathcal{S}_q(n)$. Schur-Weyl duality relates q -Schur algebras, and Hecke algebras $\mathcal{H}_q(\Sigma_r)$ associated to symmetric groups Σ_r . We view this phenomenon in chapter one, and its relevance for the representation theory of finite general linear groups $GL_n(\mathbb{F}_q)$ over a field k of characteristic l , coprime to q . Parametrizations of irreducible representations for all these algebras are given. We assemble a few facts concerning wreath products of algebras.

Chapter two opens with a description of the abacus presentation of partitions, due to G. James. We recall Nakayama's parametrization of blocks $k\mathbf{B}_{\tau,w}^\Sigma$ of symmetric groups by their l -core τ and their weight w , as well as parametrizations of blocks $k\mathbf{B}_{\tau,w}^{\mathcal{S}_q}$ of q -Schur algebras, blocks $k\mathbf{B}_{\tau,w}^{\mathcal{H}_q}$ of Hecke algebras, and unipotent blocks $k\mathbf{B}_{\tau,w}^{G_q}$ of finite general linear groups. Blocks $k\mathbf{B}_{\tau,w}^\Sigma$ have abelian defect groups if, and only if, $w < l$. Chuang and Rouquier's general theory of \mathfrak{sl}_2 -categorification implies that $k\mathbf{B}_{\tau,w}^\mathcal{X}$ is derived equivalent to $k\mathbf{B}_{\tau',w}^\mathcal{X}$, for a fixed weight w , fixed $\mathcal{X} \in \{\Sigma, \mathcal{S}_q, \mathcal{H}_q, G_q\}$, and various τ, τ' .

Blocks of symmetric groups are in one-one correspondence with weight spaces in the basic representation of $\widehat{\mathfrak{sl}_p}$. Rock blocks are a particularly symmetric class of blocks, distinguished combinatorially via their abacus presentation. When $w < l$, Chuang and Kessar's structure theorem states that a Rock block $k\mathbf{B}_{\rho,w}^\Sigma$ of weight w is Morita equivalent to a wreath product $k\mathbf{B}_{\emptyset,1}^\Sigma \wr \Sigma_w$ of a cyclic defect block $k\mathbf{B}_{\emptyset,1}^\Sigma$, and a symmetric group Σ_w on w letters. A stated corollary of this theorem is a formula for the decomposition matrix of a symmetric group Rock block of abelian defect discovered by Chuang and K.M. Tan, and independently by B. Leclerc and H. Miyachi:

$$d_{\underline{\lambda}\underline{\mu}} = \sum_{\alpha \in \Lambda_w^{p+1}, \beta \in \Lambda_w^p} \prod_{j=0}^{p-1} c(\lambda^j; \alpha^j, \beta^j) c(\mu^j; \beta^j, (\alpha^{j+1})').$$

In this formula, $\underline{\lambda} = [\lambda^0, \dots, \lambda^{l-1}]$ is the p -quotient of a partition with p -core ρ , relative to a certain abacus presentation; the numbers $c(\lambda; \mu, \nu)$ are Littlewood-Richardson coefficients.

We state R. Paget's description of the Mullineux map on a Rock block of a Hecke algebra. We proceed to describe Brauer correspondence for blocks of finite general linear groups, and then for blocks of symmetric groups. This chapter ends with the statement of a criterion of Broué, for the lifting of a character correspondence between symmetric algebras to a Morita equivalence.

The first honest mathematics appears in chapter three. We sketch a proof, à la Chuang-Kessar, of a structure theorem for Rock blocks of finite general linear groups of abelian defect. More precisely, we show that a Rock block $k\mathbf{B}_{\rho,w}^{G_q}$ is Morita equivalent to the wreath product $k\mathbf{B}_{\emptyset,1}^{G_q} \wr \Sigma_w$, when $w < l$ (theorem 72). A simple corollary is the Morita equivalence of $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ and $k\mathbf{B}_{\emptyset,1}^{\mathcal{H}_q} \wr \Sigma_w$, so long as $w < l$, and $q \neq 1$ modulo l (theorem 80).

This structure theorem for Rock blocks of finite general linear groups of abelian defect appeared in a previous paper of mine, and was written down independently by Miyachi [57]. An application given in my paper is the revelation of Morita equivalences between weight two blocks of finite general linear groups $GL_n(\mathbb{F}_q)$, as q varies. Thanks to Chuang and Rouquier's theory it must now be possible to generalise this result, and give comparisons between abelian defect blocks of $GL_n(\mathbb{F}_q)$ of arbitrary weight, as q varies. We choose not to spend time chomping on this old pie, since we have become aware of dishes with a more exotic, and alluring aroma.

In chapter four, we turn to Rock blocks of symmetric groups, of non-abelian defect. We give a sweet proof that in characteristic two, once a Rock block of a symmetric group has been localised at some idempotent, Chuang and Kessar's theorem generalises to non-abelian defect. To be precise, we prove that in characteristic two, $ek\mathbf{B}_{\rho,w}^{\Sigma}e$ is Morita equivalent to $k\Sigma_2 \wr \Sigma_w$, for some idempotent $e \in k\mathbf{B}_{\rho,w}^{\Sigma}$ (theorem 84). Our proof involves the Brauer morphism. This idea can be contorted and extended, to give a result in arbitrary characteristic. Indeed, to any Rock block $k\mathbf{B}_{\rho,w}^{\Sigma}$, we associate a natural l -permutation module kM , whose endomorphism ring $k\mathcal{E}$ is Morita equivalent to $k\Sigma_w$ (theorem 86).

Schur-Weyl duality for Σ_w is naturally visible inside a Rock block of weight w , via the Brauer homomorphism, and our fifth chapter is dedicated to establishing this truth. Formally, we prove that $k\mathbf{B}_{\rho,w}^{\Sigma}/\text{Ann}(kM)$ is Morita equivalent to the Schur algebra $S(w, w)$, and that the $S(w, w)$ - $k\Sigma_w$ -bimodule corresponding to the $k\mathbf{B}_{\rho,w}^{\Sigma}$ - $k\mathcal{E}$ -bimodule kM is twisted tensor space (theorem 90).

Chapter six begins with a criterion for the lifting of a character correspondence between quasi-hereditary algebras to a Ringel duality (theorem 109), echoing Broué's criterion for a Morita equivalence between symmetric algebras. We use this result to prove the existence of Ringel dualities between certain subquotients $kA_{(0,a_1,\dots,a_{p-2})}$ and $kB_{(a_1,\dots,a_{p-2},0)}$ of $k\mathbf{B}_{\rho,w}^{\Sigma}$ (theorem 123). We call this collection of Ringel dualities a “walk along

the abacus”, because it is reminiscent of J. A. Green’s observations on the homological algebra of the Brauer tree [37].

In the seventh chapter, we introduce the James adjustment algebra of a Hecke algebra block. This is a quotient of the block by a nilpotent ideal, whose decomposition matrix is equal to the James adjustment matrix of the block. The principal result of this chapter is theorem 132, which states that the James adjustment algebra of $k\mathbf{B}_{\rho,w}^\Sigma$ is Morita equivalent to a direct sum,

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right),$$

of tensor products of Schur algebras. The decomposition matrix of $k\mathbf{B}_{\rho,w}^\Sigma$ is a product of the matrix of sums of products of Littlewood Richardson coefficients defined by Chuang & Tan, and Leclerc & Miyachi, and the decomposition matrix of the James adjustment algebra.

At the entrance to chapter eight, we define a novel double construction. Indeed, given a bialgebra B equipped with an algebra anti-endomorphism σ , which is also a coalgebra anti-endomorphism, and a dual bialgebra B^* , we project the structure of an associative algebra onto $B \otimes B^*$ (theorem 138). Particular examples of these doubles show a remarkably close resemblance to Rock blocks.

If Q is a quiver, let P_Q be the path super-algebra of Q , modulo all quadratic relations. Let $P_Q(n)$ be the super-algebra Morita equivalent to P_Q , all of whose irreducible modules have dimension n . The coordinate ring of $P_Q(n)$ is a super-bialgebra, whose double we denote $\mathcal{D}_Q(n)$. We call such an algebra a Schur quiver double, or Schiver double. We prove that $\mathcal{D}_Q(n)$ is independent of the orientation of Q (theorem 157). Conjecture 165 predicts that the Rock block $k\mathbf{B}_{\rho,w}^\Sigma$ is Morita equivalent to a summand $\mathcal{D}_{A_{l-1}}(w, w)$ of the Schiver double $\mathcal{D}_{A_{l-1}}(w)$ associated to a quiver of type A_{l-1} .

The principal obstacle to a proof of this conjecture via the methods introduced here, has been my impotence in producing a suitable grading on a Rock block. The Schiver doubles are naturally $\mathbb{Z}_+ \times \mathbb{Z}_+$ -graded algebras, the degree zero part being isomorphic to a tensor product

$$\bigotimes_{v \in V(Q)} \mathcal{S}_v(n)$$

of classical Schur algebras (remark 156). Such gradings on the Rock blocks remain elusive. Towards the end of chapter 8, we sketch a proof that the

graded ring associated to a certain filtration on a Rock block, resembles an algebra summand of a Schiver double.

If the gradings proposed here on the Rock blocks do exist, then they will pass non-canonically via derived equivalences to $\mathbb{Z} \times \mathbb{Z}$ -gradings on arbitrary blocks [66]. Is it true that every symmetric group block can be positively graded, so that the degree zero part is isomorphic to the James adjustment algebra of the block ?

In the ninth chapter we continue the study of Schiver doubles. Given any vertex v of the quiver Q with an arrow a emanating from it, we define a non-trivial complex $\mathcal{P}_r(a)$ for $\mathcal{D}_Q(n, r)$ whose homology groups are all $\mathcal{S}_v(n, r)$ modules, and whose character is the power sum p_r (theorem 168).

In the tenth and final chapter of this article, we consider Schiver doubles associated to quivers of type A_∞ , which enjoy a number of special homological properties. So long as $n \geq r$, the module category of $\mathcal{D}_{A_\infty}(n, r)$ is a highest weight category, and Ringel self-dual (theorem 175). We speculate that any Rock block $k\mathbf{B}_{\rho, w}^{S_q}$ of a q -Schur algebra is Morita equivalent to a certain subquotient of $\mathcal{D}_{A_\infty}(w, w)$ (conjecture 178). We prove the existence of a long exact sequence of $\mathcal{D}_{A_\infty}(n, r)$ -modules

$$\dots\dots\dots \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \dots\dots\dots,$$

which generalises Green's walk along the Brauer tree for an infinite Brauer line. (theorem 182)

Chapters one and two contain introductory material. Most of it should be familiar to students of symmetric groups, q -Schur algebras, or the like. General aspects of finite group representation theory such as Brauer correspondence are often omitted from presentations of type A representation theory, but we include a brief account of this correspondence here. I consider this to be important philosophically, as well as being necessary for some of our proofs. This article was conceived before the fire of modular representation theory laid by R. Brauer, and his vision of a sympathy between global and local representations is bred in its bones.

The third, fourth, and fifth chapters all make use of local representation theory, and should properly be read consecutively. The appearance of Schur algebras in chapter five should not be a great surprise to students of semisimple algebraic groups familiar with Steinberg's tensor product theorem. However, I hope our approach via the Brauer morphism is at least provocative: it is so far unclear how to interpret J. Alperin's conjecture homologically in nonabelian defect.

Chapter six can be read independently of chapters three to five, and rests on the theory of quasi-hereditary algebras. With R. Paget's description of the Mullineux map on a Hecke algebra Rock block we cobble a pair of shiny black boots; wearing these we are able to comfortably walk along the abacus.

The description of the James adjustment algebra of a symmetric group Rock block in chapter seven relies on all the theory developed in earlier chapters. The results of chapter three allow one to understand some aspects of Rock blocks of Hecke algebras in characteristic zero, at a root of unity. The Schur algebra quotient of chapter five provides information which can be carried across the abacus using the Ringel dualities of chapter six.

Beyond this complex crescendo come chapters eight, nine, and ten. The conjectures made here concerning Rock blocks appear to be quite deep, and if proved, would envelop all the results of earlier chapters. However, their presentation is logically independent of chapters three to seven, and carries a lighter burden of notation.

The development of the article is in the direction of Time's arrow, so that more recent ideas appear towards the end of the monograph.

I am most grateful to Joe Chuang, to Karin Erdmann, and to Rowena Paget, for encouraging me amongst these ideas, and to Steffen Koenig. Hannah Turner supported me financially (partly), and libidinally (entirely). The E.P.S.R.C. gave me some money, as well. I thank the referee, for his careful reading of the manuscript, and useful comments.

This work, its morality, and the wilful emotions which dominated its creation, are dedicated to Joe Silk. He was a dear, demonic friend to me, and I wish ... to wish him farewell.

Chapter I

Highest weight categories, q -Schur algebras, Hecke algebras, and finite general linear groups.

We brutally summarise the representation theory of the q -Schur algebra, of Hecke algebras of type A, and of finite general linear groups in non-describing characteristic.

Although in later chapters, we will invoke such theory over more general commutative rings, for simplicity of presentation, in this chapter we only consider representation theory over a field k , of characteristic l .

Highest weight categories.

We state some of the principal definitions and results of E. Cline, B. Parshall, and L. Scott's paper, [17].

Definition 1 ([17], 3.1) *Let \mathcal{C} be a locally Artinian, Abelian category over k , with enough injectives. Let Λ be a partially ordered set, such that every interval $[\lambda, \mu]$ is finite, for $\lambda, \mu \in \Lambda$. The category \mathcal{C} is a highest weight category with respect to Λ if,*

(a) Λ indexes a complete collection $\{L(\lambda)\}_{\lambda \in \Lambda}$ of non-isomorphic simple objects of \mathcal{C} .

(b) Λ indexes a collection $\{\nabla(\lambda)\}_{\lambda \in \Lambda}$ of “costandard objects” of \mathcal{C} , for each of which there exists an embedding $L(\lambda) \hookrightarrow \nabla(\lambda)$, such that all composition factors $L(\mu)$ of $\nabla(\lambda)/L(\lambda)$ satisfy $\mu < \lambda$.

(c) For $\lambda, \mu \in \Lambda$, we have, $\dim_k \text{Hom}(\nabla(\lambda), \nabla(\mu)) < \infty$, and in addition, $[\nabla(\lambda) : L(\mu)] < \infty$.

(d) An injective envelope $I(\lambda) \in \mathcal{C}$ of $L(\lambda)$ possesses a filtration

$$0 = F_0(\lambda) \subset F_1(\lambda) \subset \dots,$$

such that,

(i) $F_1(\lambda) \cong \nabla(\lambda)$.

(ii) For $n > 1$, we have $F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu)$, for some $\mu = \mu(n) > \lambda$.

(iii) For $\mu \in \Lambda$, we have $\mu(n) = \mu$ for finitely many n .

(iv) $I(\lambda) = \bigcup_i F_i(\lambda)$.

Definition 2 ([17], 3.6) *Let S be a finite dimensional algebra over k . Then S is said to be quasi-hereditary if the category $S - \text{mod}$, of finitely generated left S -modules is a highest weight category.*

For $M \in \mathcal{C}$, and $\Gamma \subset \Lambda$, let M_Γ be the largest subobject of M , all of whose composition factors $L(\gamma)$ correspond to elements $\gamma \in \Gamma$.

Theorem 3 ([17], 3.5) *Let \mathcal{C} be a highest weight category with respect to Λ . Let $\Gamma \subset \Lambda$ be a finitely generated ideal, and let $\Omega \subset \Lambda$ be a finitely generated coideal. Suppose that $\Gamma \cap \Omega$ is a finite set.*

There exists a quasi-hereditary algebra $S(\Gamma \cap \Omega)$ with poset $\Gamma \cap \Omega$, unique up to Morita equivalence, such that the derived category $D^b(S(\Gamma \cap \Omega) - \text{mod})$ may be identified as the full subcategory of $D^b(\mathcal{C})$ represented as complexes of finite sums of modules $I(\omega)_\Gamma$, with $\omega \in \Gamma \cap \Omega$. \square

Remark 4 If S is a quasi-hereditary algebra with module category \mathcal{C} , under the hypotheses of theorem 3 we may choose $S(\Gamma \cap \Omega)$ to be a subquotient $i(S/SjS)i$ of S , where i, j are certain idempotents in S .

Theorem 5 ([17], 3.4, 3.11, [16], 4.3b) *Let S be a quasi-hereditary algebra, with respect to a poset Λ . Then,*

(a) Λ indexes a collection $\{\Delta(\lambda)\}_{\lambda \in \Lambda}$ of “standard objects” of \mathcal{C} , for each of which there exists a surjection $\phi_\lambda : \Delta(\lambda) \rightarrow L(\lambda)$, such that all composition factors $L(\mu)$ of $\ker(\phi_\lambda)$ satisfy $\mu < \lambda$.

(b) The projective cover $P(\lambda)$ of $L(\lambda)$ possesses a filtration,

$$P(\lambda) = G_0(\lambda) \supset G_1(\lambda) \supset \dots \supset G_N(\lambda) = 0,$$

such that,

(i) $G_0(\lambda)/G_1(\lambda) \cong \Delta(\lambda)$.

(ii) For $n > 0$, we have $G_n(\lambda)/G_{n+1}(\lambda) \cong \Delta(\mu)$, for some $\mu > \lambda$.

(c) For projective objects P in \mathcal{C} , the number $[P : \Delta(\lambda)]$ of objects $\Delta(\lambda)$ appearing in a filtration by standard objects, is independent of filtration, for $\lambda \in \Lambda$.

For injective objects I in \mathcal{C} , the number $[I : \nabla(\lambda)]$ of objects $\nabla(\lambda)$ appearing in a filtration by costandard objects, is independent of filtration, for $\lambda \in \Lambda$.

(d) $[I(\mu) : \nabla(\lambda)] = [\Delta(\lambda) : L(\mu)]$, for $\lambda, \mu \in \Lambda$.

Dually, $[P(\mu) : \Delta(\lambda)] = [\nabla(\lambda) : L(\mu)]$, for $\lambda, \mu \in \Lambda$.

(e) The category $\text{mod} - S$ of right modules over S is a highest weight category. \square

Remark 6 Let S be a finite dimensional algebra, whose simple modules $\{L(\lambda)\}_{\lambda \in \Lambda}$ are parametrised by a poset Λ . Suppose that S satisfies conditions (a) and (b) of theorem 5. Then S is quasi-hereditary with respect to Λ , by a dual to theorem 5.

Definition 7 The decomposition matrix of a quasi-hereditary algebra S is the matrix $(d_{\lambda\mu})$ of composition multiplicities $([\Delta(\lambda) : L(\mu)])$, whose rows and columns are indexed by Λ .

Let S be a quasi-hereditary algebra with respect to Λ . We describe some elements of the theory of tilting modules for S , due to C. Ringel [63] (see also [27], A4).

Definition 8 A tilting module for S is a finite dimensional S -module, which may be filtered by standard modules, and may also be filtered by co-standard modules.

Theorem 9 ([27], A4, theorem 1) For $\lambda \in \Lambda$, there is an indecomposable tilting module $T(\lambda)$, unique up to isomorphism, such that $[T(\lambda) : L(\lambda)] = 1$, and all composition factors $L(\mu)$ of $T(\lambda)$ satisfy $\mu \leq \lambda$.

Every tilting module for S is a direct sum of modules $T(\lambda)$, $\lambda \in \Lambda$. \square

Definition 10 A full tilting module for S is a tilting module, in which every $T(\lambda)$ occurs as a direct summand.

A Ringel dual S' of S is defined to be $\text{End}_S(T)^{op}$, where T is a full tilting module for S .

Remark 11 The Ringel dual of S is unique, up to Morita equivalence.

We say the bimodule ${}_S T_{S'^{op}}$ defines a Ringel duality between S, S' .

Theorem 12 ([27], A4, theorem 2) *The Ringel duals S' of S are quasi-hereditary algebras, with respect to the poset Λ^{op} , opposite to Λ .*

The standard S' -module corresponding to $\lambda \in \Lambda^{op}$ is given by

$$\Delta'(\lambda) = \text{Hom}_S(T, \nabla(\lambda)).$$

Dually, the costandard S' -module corresponding to $\lambda \in \Lambda^{op}$ is given by

$$\nabla'(\lambda) = \Delta^r(\lambda) \otimes_S T,$$

where $\Delta(\lambda)^r$ denotes the standard right module for S . \square

q -Schur algebras.

S. Donkin, and R. Dipper have associated, to a natural number n , and a non-zero element $q \in k$, a bialgebra $\mathcal{A}_q(n)$ [23]. This bialgebra is a q -deformation of the bialgebra $\mathcal{A}(n) = k[M]$ of regular functions on the associative algebra $M = M_n(k)$, of $n \times n$ matrices over k . Whilst the undeformed bialgebra $\mathcal{A}(n) = \mathcal{A}_1(n)$ is commutative, $\mathcal{A}_q(n)$ is noncommutative, for $q \neq 1$.

In general, $\mathcal{A}_q(n)$ may be decomposed by degree as a direct sum,

$$\mathcal{A}_q(n) = \bigoplus_{r \geq 0} \mathcal{A}_q(n, r)$$

of finite dimensional coalgebras. Thus, upon writing $\mathcal{S}_q(n, r)$ for $\mathcal{A}_q(n, r)^*$, the bialgebra $\mathcal{S}_q(n)$, defined to be the graded dual of $\mathcal{A}_q(n)$, decomposes as a direct sum,

$$\mathcal{S}_q(n) = \bigoplus_{r \geq 0} \mathcal{S}_q(n, r)$$

of finite dimensional algebras. The algebras $\mathcal{S}_q(n, r)$ are the q -Schur algebras, first introduced by R. Dipper and G. James [26].

Let $\Lambda(n, r)$ be the poset of partitions of r with n parts or fewer, with the dominance ordering \trianglelefteq .

Theorem 13 ([27], 0.22) *The q -Schur algebra $\mathcal{S}_q(n, r)$ is a highest weight category with respect to $\Lambda(n, r)$. \square*

Let p be the least natural number such that $1 + q + \dots + q^{p-1} = 0$, if such exists. Otherwise, let $p = \infty$.

Theorem 14 ([27], 4.3(7)) *If $p = \infty$, then $\mathcal{S}_q(n, r)$ is semisimple for all n, r . In this case, $\Delta(\lambda) \cong L(\lambda)$, for all $\lambda \in \Lambda(n, r)$.*

Remark 15 The algebra $\mathcal{S}_q(n, r)$ possesses a natural anti-automorphism σ , inherited from the transpose operation on the matrix algebra M ([27], 4.1).

It is thus possible to twist a left/right $\mathcal{S}_q(n, r)$ -module by σ , to obtain a right/left $\mathcal{S}_q(n, r)$ -module. Composing this twist with the duality functor, we may associate to a left/right $\mathcal{S}_q(n, r)$ -module N , a left/right $\mathcal{S}_q(n, r)$ -module N^* , its *contragredient dual*.

The contragredient dual of a standard module $\Delta(\lambda)$ is isomorphic to the costandard module, $\nabla(\lambda)$. The contragredient dual of a costandard module $\nabla(\lambda)$ is isomorphic to the standard module, $\Delta(\lambda)$.

Remark 16 In general, $\mathcal{S}_q(n, 1)$ is isomorphic to the matrix algebra M , and therefore has a unique irreducible left module E , and a unique irreducible right module E^{op} .

S. Donkin has defined the q -exterior powers $\bigwedge_q^r E$ (respectively $\bigwedge_q^r E^{op}$) of E (respectively E^{op}), which are left (respectively right) $\mathcal{S}_q(n, r)$ -modules of dimension $\binom{n}{r}$, exchanged under the anti-automorphism σ ([27], 1.2). He has also defined q -exterior powers $\bigwedge_q^r M$ of M , which are $\mathcal{S}_q(n, r)$ - $\mathcal{S}_q(n, r)$ -bimodules of dimension $\binom{n^2}{r}$ ([27], 4.1).

For a sequence $\alpha = (\alpha_1, \dots, \alpha_m)$ of natural numbers, whose sum is r , let us define

$$\bigwedge_q^\alpha E = \bigwedge_q^{\alpha_1} E \otimes \dots \otimes \bigwedge_q^{\alpha_m} E.$$

$$\bigwedge_q^\alpha E^{op} = \bigwedge_q^{\alpha_1} E^{op} \otimes \dots \otimes \bigwedge_q^{\alpha_m} E^{op}.$$

Theorem 17 (*S. Donkin, [27], 1.2, 4.1*)

(a) For $\tau \in \Sigma_m = \text{Sym}\{1, \dots, m\}$, there is an $\mathcal{S}_q(n, r)$ -module isomorphism between the exterior powers, $\bigwedge_q^{(\alpha_1, \dots, \alpha_m)} E$, and $\bigwedge_q^{(\alpha_{\tau 1}, \dots, \alpha_{\tau m})} E$.

(b) There is a non-degenerate bilinear form,

$$\langle, \rangle: \bigwedge_q^\alpha E^{op} \times \bigwedge_q^\alpha E \rightarrow k,$$

such that $\langle x \circ s, y \rangle = \langle x, s \circ y \rangle$. Therefore, $\bigwedge_q^\alpha E \cong (\bigwedge_q^\alpha E)^*$, as left $\mathcal{S}_q(n, r)$ -modules, and $\bigwedge_q^\alpha E^{op} \cong (\bigwedge_q^\alpha E^{op})^*$, as right $\mathcal{S}_q(n, r)$ -modules.

(c) Direct summands of q -exterior powers $\bigwedge_q^\alpha E$ are tilting modules for $\mathcal{S}_q(n, r)$. The restriction of $\bigwedge_q^r M$ to a left $\mathcal{S}_q(n, r)$ -module is a full tilting module. The restriction of $\bigwedge_q^r M$ to a right $\mathcal{S}_q(n, r)$ -module is also a full tilting module.

(d) Let $n \geq r$. The $\mathcal{S}_q(n, r)$ - $\mathcal{S}_q(n, r)$ -bimodule $\bigwedge_q^r M$ defines a Ringel duality between $\mathcal{S}_q(n, r)$ and $\mathcal{S}_q(n, r)^{op} = \mathcal{S}_q(n, r)$.

Hecke algebras associated to symmetric groups.

Let $\Sigma_r = \text{Sym}\{1, 2, \dots, r\}$ be the symmetric group on r letters.

Definition 18 The Hecke algebra $\mathcal{H}_q(\Sigma_r)$ associated to Σ_r , is the associative k -algebra with generators

$$\{T_i | i = 1, \dots, r-1\},$$

subject to the relations,

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i = 1, \dots, r-2,$$

$$T_i T_j = T_j T_i, \quad |j - i| > 1,$$

$$(T_i - q)(T_i + 1) = 0, \quad i = 1, \dots, r-1.$$

When $q = 1$, the Hecke algebra $\mathcal{H}_q(\Sigma_r)$ is isomorphic to the group algebra $k\Sigma_r$. Many of the properties of $k\Sigma_r$ generalise to $\mathcal{H}_q(\Sigma_r)$. For example, we have the following theorem.

Theorem 19 ([44], chapter 7)

(a) $\mathcal{H}_q(\Sigma_r)$ possesses an outer automorphism $\#$, the “sign automorphism”, which takes T_i to $-T_i + q - 1$.

(b) Given $w \in \Sigma_r$, and a reduced expression $w = s_1 \dots s_{l(w)}$ as a product simple transpositions $s_j \in \{(i, i+1), 1 \leq i \leq r-1\}$, we may define an element $T_w = T_{s_1} \dots T_{s_{l(w)}} \in \mathcal{H}_q(\Sigma_r)$, where $T_{(i, i+1)} = T_i$. The element T_w is independent of the choice of reduced expression.

(c) The set $\{T_w | w \in \Sigma_r\}$ is a basis for $\mathcal{H}_q(\Sigma_r)$. \square

Remark 20 There is a one dimensional “trivial module” for $\mathcal{H}_q(\Sigma_r)$, denoted k , on which T_w acts as $q^{l(w)}$. The twist of the trivial module by $\#$ is the “sign module”, denoted sgn , on which T_w acts as $(-1)^{l(w)}$.

There is an elementary relation between Hecke algebras, and q -Schur algebras, which is commonly exploited to describe the representation theory of the Hecke algebra. A proof, with references to various sources, is given in [50], 1.2.

Theorem 21 “Schur-Weyl duality”

- (a) The $\mathcal{S}_q(n, r)$ -module $E^{\otimes r}$ is a tilting module.
- (b) There is an algebra surjection $\mathcal{H}_q(\Sigma_r) \twoheadrightarrow \text{End}_{\mathcal{S}_q(n, r)}(E^{\otimes r})$.
- (c) $\mathcal{S}_q(n, r) = \text{End}_{\mathcal{H}_q(\Sigma_r)}(E^{\otimes r})$.

Remark 22 If λ is a partition, then λ' denotes the conjugate partition.

A partition λ is said to be p -regular if, and only if, λ does not have $\geq p$ identical parts.

A partition λ is said to be p -restricted if, and only if, λ' is p -regular.

The $\mathcal{S}_q(n, r)$ - $\mathcal{H}_q(\Sigma_r)$ -bimodule $E^{\otimes r}$ is often referred to as *tensor space*. If $n \geq r$, then this bimodule is particularly regular.

Theorem 23 (J.A. Green [38], S. Donkin, [27], 2.1, 4.4) Let $n \geq r$.

- (a) There is an idempotent $\xi_\omega \in \mathcal{S}_q(n, r)$, such that

$$E^{\otimes r} \cong \mathcal{S}_q(n, r)\xi_\omega,$$

as $\mathcal{S}_q(n, r)$ -modules.

- (b) The $\mathcal{S}_q(n, r)$ -module $E^{\otimes r}$ is projective, and injective.
- (c) $\mathcal{H}_q(\Sigma_r) = \text{End}_{\mathcal{S}_q(n, r)}(E^{\otimes r}) = \xi_\omega \mathcal{S}_q(n, r) \xi_\omega$.
- (d) The Schur functor,

$$\text{Hom}_{\mathcal{S}_q(n, r)}(E^{\otimes r}, -) : \mathcal{S}_q(n, r) - \text{mod} \rightarrow \mathcal{H}_q(\Sigma_r) - \text{mod},$$

is exact, and takes simple modules either to simple modules, or to zero. We obtain all simple $\mathcal{H}_q(\Sigma_r)$ -modules from the set of simple $\mathcal{S}_q(n, r)$ -modules, in this way.

- (e) If λ is a p -nonrestricted partition, then $\text{Hom}_{\mathcal{S}_q(n, r)}(E^{\otimes r}, L(\lambda)) = 0$. Otherwise, if λ is a p -restricted partition, then $D_\lambda = \text{Hom}_{\mathcal{S}_q(n, r)}(E^{\otimes r}, L(\lambda))$ is a simple $\mathcal{H}_q(\Sigma_r)$ -module.

(f) The set,

$$\{D_\lambda \mid \lambda \text{ a } p\text{-restricted partition of } r\},$$

is a complete set of non-isomorphic irreducible $\mathcal{H}_q(\Sigma_r)$ -modules. \square

Remark 24 Let $D^{\lambda'} = D_\lambda^\#$. It follows from theorem 23(f) that

$$\{D^\lambda \mid \lambda \text{ a } p\text{-regular partition of } r\},$$

is a complete set of non-isomorphic irreducible $\mathcal{H}_q(\Sigma_r)$ -modules.

Definition 25 The Specht module associated to a partition λ is the $\mathcal{H}_q(\Sigma_r)$ -module,

$$S^\lambda = \text{Hom}_{\mathcal{S}_q(r,r)}(E^{\otimes r}, \nabla(\lambda)).$$

The $\mathcal{H}_q(\Sigma_r)$ -module S_λ is defined to be,

$$S_\lambda = \text{Hom}_{\mathcal{S}_q(r,r)}(E^{\otimes r}, \Delta(\lambda)).$$

The decomposition matrix of $\mathcal{H}_q(\Sigma_r)$ is the matrix $(d_{\lambda\mu})$ of composition multiplicities $[S^\lambda : D^\mu]$, indexed by partitions λ of r , and p -regular partitions μ of r .

Remark 26

(a) It follows from theorem 23 that $S^{\lambda\#} \cong S_{\lambda'} \cong S^{\lambda'*}$. We call S_λ a dual Specht module.

(b) When $q = 1$, and k is a field of characteristic zero, the Specht modules S^λ form a complete set of non-isomorphic simple $k\Sigma_r$ -modules. We write χ^λ for the irreducible character of Σ_r corresponding to S^λ .

(c) The standard $\mathcal{S}_q(r, r)$ module $\Delta((1^r)) = \bigwedge^r(E)$ is one dimensional (the "determinant representation"). The Specht modules $S^{(r)}$ and $S^{(1^r)}$ are also one dimensional (the "trivial representation", and the "sign representation").

Definition 27 The Young module associated to a partition λ is the $\mathcal{H}_q(\Sigma_r)$ -module

$$Y^\lambda = \text{Hom}_{\mathcal{S}_q(r,r)}(E^{\otimes r}, I(\lambda)),$$

where $I(\lambda)$ is the injective hull of $L(\lambda)$. The twisted Young module associated to λ is the $\mathcal{H}_q(\Sigma_r)$ -module $Y^{\lambda\#}$.

Remark 28

(a) By Schur-Weyl duality, Y^λ is an indecomposable $\mathcal{H}_q(\Sigma_r)$ -module, with a filtration by Specht modules S^μ , with $\mu \supseteq \lambda$, and a single section S^λ . Thus, $Y^{\lambda\#}$ is an indecomposable $\mathcal{H}_q(\Sigma_r)$ -module, filtered by dual Specht modules S_μ , with $\mu \supseteq \lambda'$, and a single section $S_{\lambda'}$.

(b) It follows from theorem 23 that Y^λ is projective if, and only if, λ is p -restricted.

(c) A partition $\lambda = (\lambda_i)$ of r defines a Young subgroup $\Sigma_\lambda = \times_i \Sigma_{\lambda_i}$ of Σ_r . Inducing the trivial representation from $\mathcal{H}_q(\Sigma_\lambda) = \otimes_i \mathcal{H}_q(\Sigma_{\lambda_i})$ up to $\mathcal{H}_q(\Sigma_r)$, we obtain a module M^λ , which is a direct summand of tensor space, as a $\mathcal{H}_q(\Sigma_r)$ -module. This is because induction from Young subalgebras corresponds to taking tensor products, under Schur-Weyl duality. It follows that the module M^λ is a direct sum of Young modules. In a direct sum decomposition, M^λ has a unique indecomposable summand isomorphic to Y^λ , with all other summands isomorphic to Y^μ , with $\mu \supseteq \lambda$.

(d) As a $\mathcal{H}_q(\Sigma_r)$ -module, tensor space is isomorphic to a direct sum of Young modules.

Finite general linear groups.

Throughout this section, q is a prime power, coprime to l . We consider representations of the finite general linear group $GL_n(q)$, over the field k .

Let V be an n dimensional vector space over \mathbb{F}_q . Let $\underline{n} = (n_1, \dots, n_N)$ be a sequence of natural numbers whose sum is n . Let us fix an \mathbb{F}_q -basis of V . We define a collection of subgroups of $GL(V) = GL_n(\mathbb{F}_q) = GL_n(q)$ relative to this basis:

A *maximal torus* $T(q)$ - the subgroup of diagonal matrices.

A *Borel subgroup* $B(q)$ - the subgroup of upper triangular matrices.

A *Levi subgroup* $L_{\underline{n}}(q)$ - viewed as,

$$\begin{pmatrix} GL_{n_1}(q) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & GL_{n_N}(q) \end{pmatrix}$$

A *parabolic subgroup* $P_{\underline{n}}(q)$ - viewed as,

$$\begin{pmatrix} GL_{n_1}(q) & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & GL_{n_N}(q) \end{pmatrix}$$

A unipotent subgroup $U_{\underline{n}}(q)$ - viewed as,

$$\begin{pmatrix} 1_{n_1} & * & \dots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1_{n_N} \end{pmatrix}$$

The Weyl group $W \cong \Sigma_n$ - permutation matrices. The following lemma is well known.

Lemma 29 (a) The normaliser of $U_{\underline{n}}(q)$ is $P_{\underline{n}}(q) = U_{\underline{n}}(q) \rtimes L_{\underline{n}}(q)$.

(b) We have Bruhat decomposition,

$$GL_n(q) = \coprod_{w \in W} B(q)wB(q). \square$$

Definition 30 Harish-Chandra induction is the functor,

$$HCInd_{L_{\underline{n}}(q)}^{GL_n(q)} : kL_{\underline{n}}(q) - mod \rightarrow kGL_n(q) - mod,$$

$$M \mapsto k[GL_n(q)/U_{\underline{n}}(q)] \bigotimes_{L_{\underline{n}}(q)} M.$$

Harish-Chandra restriction is the functor,

$$HCRes_{L_{\underline{n}}(q)}^{GL_n(q)} : GL_n(q) - mod \rightarrow L_{\underline{n}}(q) - mod,$$

$$M \mapsto k[U_{\underline{n}}(q) \backslash GL_n(q)] \bigotimes_{GL_n(q)} M.$$

Remark 31 Because q and l are coprime, the $kGL_n(q)$ - $kL_{\underline{n}}(q)$ bimodule

$$k[GL_n(q)/U_{\underline{n}}(q)],$$

is a summand of the ordinary induction bimodule ${}_{GL_n(q)}k[GL_n(q)]_{L_{\underline{n}}(q)}$, and so Harish-Chandra induction is exact. Its left and right adjoint, Harish-Chandra restriction, is also exact.

Assuming the notation of J. Brundan, R. Dipper, and A. Kleshchev's article [10], we introduce a set to parametrise conjugacy classes in $GL_n(q)$.

For $\sigma \in \bar{\mathbb{F}}_q^*$ of degree d_σ over \mathbb{F}_q , let (σ) be the companion matrix representing σ in $GL_{d_\sigma}(q)$.

For a natural number k , let

$$(\sigma)^k = \text{diag}((\sigma), \dots, (\sigma)) = \begin{pmatrix} (\sigma) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (\sigma) \end{pmatrix}$$

be the block diagonal matrix embedding k copies of (σ) in $GL_{d_\sigma k}(q)$.

For $\sigma, \tau \in \bar{\mathbb{F}}_q^*$ of degree d over \mathbb{F}_q , let us write $\sigma \sim \tau$ if σ and τ have the same minimal polynomial over \mathbb{F}_q .

Definition 32 (a) Let \mathcal{C}_{ss}^{pre} be the set,

$$\{((\sigma_1)^{k_1}, \dots, (\sigma_a)^{k_a}) \mid \sigma_i \in \bar{\mathbb{F}}_q^*, \sigma_i \sim \sigma_j \text{ for } i \neq j, \sum d_{\sigma_i} k_i = n\}.$$

(b) Let \sim denote an equivalence relation on \mathcal{C}_{ss}^{pre} , given by

$$((\sigma_1)^{k_1}, \dots, (\sigma_a)^{k_a}) \sim ((\tau_1)^{m_1}, \dots, (\tau_a)^{m_b})$$

so long as,

(i) $a = b$,

(ii) there exists $w \in \Sigma_a$, such that $k_{wi} = m_i$, and $\sigma_{wi} \sim \tau_i$, for $i = 1, \dots, a$.

(c) Let $\mathcal{C}_{ss} = \mathcal{C}_{ss}^{pre} / \sim$.

For $s = ((\sigma_1)^{k_1}, \dots, (\sigma_a)^{k_a}) \in \mathcal{C}_{ss}$, let $\kappa(s) = (k_1, \dots, k_a)$.

If $\underline{\lambda} = (\lambda_1, \dots, \lambda_a)$ is a sequence of partitions whose degrees are described by the sequence κ , we write $\underline{\lambda} \vdash \kappa$.

The representation theory of the principal ideal domain $\mathbb{F}_q[x]$ implies the following result.

Theorem 33 (Jordan decomposition) *The conjugacy classes of $GL_n(q)$ are in one-one correspondence with the set,*

$$\{(s, \underline{\lambda}) \mid s \in \mathcal{C}_{ss}, \underline{\lambda} \vdash \kappa(s)\}. \square$$

Theorem 34 (*J.A. Green, [36], [10], 2.3*) Fix an embedding $\bar{\mathbb{F}}_q^* \hookrightarrow \bar{\mathbb{Q}}_l^*$. There is a one-one correspondence between the set of ordinary irreducible characters of $GL_n(q)$ and the set,

$$\{(s, \underline{\lambda}) \mid s \in \mathcal{C}_{ss}, \underline{\lambda} \vdash \kappa(s)\}. \square$$

Given an embedding $\bar{\mathbb{F}}_q^* \hookrightarrow \bar{\mathbb{Q}}_l^*$, and $s \in \mathcal{C}_{ss}, \underline{\lambda} \vdash \kappa(s)$, we write $\chi(s, \underline{\lambda})$ for the irreducible character of $GL_n(q)$ corresponding to the pair $(s, \underline{\lambda})$ in the above theorem.

Remark 35 Let $s = ((\sigma_1)^{k_1}, \dots, (\sigma_a)^{k_a})$, and $\underline{\lambda} \vdash \kappa(s)$. The irreducible character $\chi(s, \underline{\lambda})$ corresponding to the pair $(s, \underline{\lambda})$, is equal to the Harish-Chandra induced character,

$$HCInd_{GL_{d\sigma_1 k_1}(q) \times \dots \times GL_{d\sigma_a k_a}(q)}^{GL_n(q)} \left(\chi((\sigma_1)^{k_1}, \lambda_1) \otimes \dots \otimes \chi((\sigma_a)^{k_a}, \lambda_a) \right).$$

Furthermore, if μ, ν are partitions of m, n , and $\sigma \in \bar{\mathbb{F}}_q^*$ has degree d , then,

$$\begin{aligned} & HCInd_{GL_{dm}(q) \times GL_{dn}(q)}^{GL_{dm+dn}(q)} (\chi((\sigma)^m, \mu) \otimes \chi((\sigma)^n, \nu)) \\ &= \sum c(\lambda; \mu, \nu) \chi((\sigma)^{m+n}, \lambda), \end{aligned}$$

where $c(\lambda; \mu, \nu)$ is the Littlewood-Richardson coefficient associated to λ, μ, ν .

Let $\mathcal{C}_{ss, l'} = \{s \in \mathcal{C}_{ss} \mid \text{the order of } s \text{ is coprime to } l'\}$.

Theorem 36 (*P. Fong and B. Srinivasan, [10], 2.4.6*) There is a decomposition of the group algebra $kGL_n(q)$ into a direct sum of two sided ideals,

$$kGL_n(q) = \bigoplus_{s \in \mathcal{C}_{ss, l'}} kB_s,$$

where the characters in B_s are given by the set,

$$\{\chi(t, \underline{\lambda}) \mid \underline{\lambda} \vdash \kappa(t), t \in \mathcal{C}_{ss} \text{ has } l\text{-regular part conjugate to } s\}. \square$$

Definition 37 The characters $\{\chi(1, \lambda) \mid \lambda \vdash n\}$ of $GL_n(q)$ are the unipotent characters.

The indecomposable components of the $GL_n(q)$ - $GL_n(q)$ -bimodule kB_1 are the unipotent blocks.

Thanks to the following theorem, we may concentrate our curiosity on unipotent blocks.

Theorem 38 (*C. Bonnafé and R. Rouquier, [5], 11.8*) Every block of $GL_n(q)$ is Morita equivalent to a unipotent block. \square

Theorem 39 (*N. Iwahori, E. Cline, B. Parshall, L. Scott, see [10], 3.2d, 3.5a*) Let $\mathcal{M} = k[GL_n(q)/B(q)]$ denote the permutation module for $kGL_n(q)$ on the coset space $GL_n(q)/B(q)$.

(a) If $q \neq 1$ modulo l , then \mathcal{M} is a projective $kGL_n(q)$ -module.

(b) \mathcal{M} is a kB_1 -module.

(c) $\text{End}_{kB_1}(\mathcal{M}) \cong \mathcal{H}_q(\Sigma_n)$. Under this isomorphism, the endomorphism defined by the double coset $B(q)wB(q)$ maps to the element T_w of $\mathcal{H}_q(\Sigma_n)$, for $w \in \Sigma_n$.

(d) The annihilator ideal $\text{Ann}_{kB_1}(\mathcal{M})$ is nilpotent.

(e) The quotient $kB_1/\text{Ann}_{kB_1}(\mathcal{M})$ is Morita equivalent to $\mathcal{S}_q(n, n)$.

The standard $\mathcal{S}_q(n, n)$ -module $\Delta(\lambda)$ corresponds, under this Morita equivalence, to a kB_1 -module, which is an l -modular reduction of the character $\chi(1, \lambda)$ of $GL_n(q)$.

(f) The $\mathcal{S}_q(n, n)$ - $\mathcal{H}_q(\Sigma_n)$ -bimodule corresponding to the kB_1 - $\mathcal{H}_q(\Sigma_n)$ -bimodule \mathcal{M} under the Morita equivalence of (d), is isomorphic to twisted tensor space $E^{\otimes n\#}$. \square

Definition 40 By theorem 39, simple kB_1 -modules are in one-one correspondence with simple modules for the q -Schur algebra, which are in one-one correspondence with the set $\Lambda(n) = \Lambda(n, n)$. We denote by $D(\lambda)$ the simple kB_1 -module corresponding to $\lambda \in \Lambda(n)$.

Remark 41 By twisted tensor space $E^{\otimes n\#}$, we mean the tensor space of theorem 21, on which the right action of $\mathcal{H}_q(\Sigma_n)$ has been twisted by the signature automorphism, $\#$.

Wreath products.

For a k -algebra A , and a natural number w , the wreath product $A \wr \Sigma_w$, is defined to be the space $A^{\otimes w} \otimes k\Sigma_w$, with associative multiplication,

$$(x_1 \otimes \dots \otimes x_w \otimes \sigma) \cdot (y_1 \otimes \dots \otimes y_w \otimes \tau) =$$

$$x_1 \cdot y_{\sigma^{-1}1} \otimes \dots \otimes x_w \cdot y_{\sigma^{-1}w} \otimes \sigma \cdot \tau,$$

for $x_1, \dots, x_w, y_1, \dots, y_w \in A$, and $\sigma, \tau \in \Sigma_w$. We collect a jumble of information concerning wreath products. One reference for such stuff is an article of J. Chuang and K.M. Tan [15].

For an A -module L , let us define the $A \wr \Sigma_w$ -module $T^{(w)}L$ to be the w -fold tensor product $L^{\otimes w}$, on which the subalgebra $A^{\otimes w}$ acts component-wise, and the symmetric group Σ_w acts by place permutations.

For an $A \wr \Sigma_w$ -module M , and a $k\Sigma_w$ -module N , we define an $A \wr \Sigma_w$ -module $M \otimes N$, by the action

$$(\alpha \otimes \sigma)(m \otimes n) = (\alpha \otimes \sigma)m \otimes \sigma n,$$

for $\alpha \in A^{\otimes w}, \sigma \in \Sigma_w, m \in M, n \in N$.

Let I be a finite set. Given a set of natural numbers $\{w_i, i \in I\}$, whose sum is w , there is a Young subgroup of Σ_w , isomorphic to $\times_{i \in I} \Sigma_{w_i}$. Accordingly, there is a subalgebra of $A \wr \Sigma_w$, isomorphic to $\bigotimes_{i \in I} A \wr \Sigma_{w_i}$.

Let Λ_l be the set of all l -regular partitions. Let $\Lambda_{l,w}^I$ be the set of I -tuples $(\lambda^i)_{i \in I}$ of l -regular partitions, whose orders $(w_i)_{i \in I}$ sum to w .

Theorem 42 (*I. MacDonald, [55]*) *Let $\{S(i), i \in I\}$ be a complete set of non-isomorphic simple A -modules. For $\underline{\lambda} \in \Lambda_{l,w}^I$, let*

$$S(\underline{\lambda}) = \text{Ind}_{\bigotimes_{i \in I} A \wr \Sigma_{w_i}}^{A \wr \Sigma_w} \left(\bigotimes_{i \in I} T^{w_i}(S(i)) \otimes D^{\lambda^i} \right).$$

The set $\{S(\underline{\lambda}) \mid \underline{\lambda} \in \Lambda_{l,w}^I\}$, is a complete set of non-isomorphic simple $A \wr \Sigma_w$ -modules. \square

Let q be a prime power, coprime to $l > 0$. Let $B(q)$ be a Borel subgroup of $GL_n(q)$. Thus, the direct product $\times^w B(q)$ of w copies of $B(q)$ is a subgroup of the base group $\times^w GL_n(q)$ inside $GL_n(q) \wr \Sigma_w$.

The following theorem is a straightforward generalisation of theorem 39.

Theorem 43 *Let \mathcal{M}_w be the $kB_1 \wr \Sigma_w$ -module $k[GL_n(q) \wr \Sigma_w / \times^w B(q)]$.*

(a) If $q \neq 1$ modulo l , then \mathcal{M}_w is a projective $kGL_n(q) \wr \Sigma_w$ -module.

(b) $\text{End}_{GL_n(q) \wr \Sigma_w}(\mathcal{M}_w) \cong \mathcal{H}_q(\Sigma_n) \wr \Sigma_w$.

(c) The simple modules sent to zero by the functor,

$$\text{Hom}_{GL_n(q) \wr \Sigma_w}(\mathcal{M}_w, -),$$

are precisely those simples $S(\underline{\lambda}), \underline{\lambda} \in \Lambda_{l,w}^{\Lambda(n)}$, for which some entry, λ^μ in $\underline{\lambda}$, is non-zero, for some p -nonrestricted partition μ . \square

Proposition 44 *The wreath product $\mathcal{H}_q(\Sigma_n) \wr \Sigma_w$ is a symmetric algebra.*

Proof:

Define a form \langle, \rangle on $\mathcal{H}_q(\Sigma_n)$, extending the following form bilinearly:

$$\langle T_u, T_v \rangle = \begin{cases} q^{l(u)} & \text{if } u = v^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

R. Dipper and G. James have proved ([25], (2.3)) that this is a symmetric, associative bilinear form. It is also non-degenerate, since q is a unit in k .

In case $n = w, q = 1$, one obtains a symmetric bilinear form \langle, \rangle_Σ on $k\Sigma_w$. Now define a bilinear form on $\mathcal{H}_q(\Sigma_n) \wr \Sigma_w$, by the formula,

$$\langle x_1 \otimes \dots \otimes x_w \otimes \sigma, y_1 \otimes \dots \otimes y_w \otimes \tau \rangle_\wr =$$

$$\langle x_1, y_{\sigma^{-1}1} \rangle \dots \langle x_w, y_{\sigma^{-1}w} \rangle \langle \sigma, \tau \rangle_\Sigma,$$

for $x_1, \dots, x_w, y_1, \dots, y_w \in \mathcal{H}_q(\Sigma_n)$, and $\sigma, \tau \in \Sigma_w$. Then \langle, \rangle_\wr is an associative, symmetric, non-degenerate bilinear form on $\mathcal{H}_q(\Sigma_n) \wr \Sigma_w$. \square

Chapter II

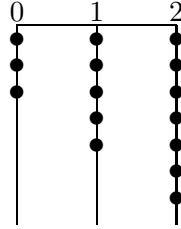
Blocks of q -Schur algebras, Hecke algebras, and finite general linear groups.

Let k be a field of characteristic l . We summarise some elements of block theory for the q -Schur algebras, the Hecke algebras, and the finite general linear groups in non-describing characteristic. We introduce the Rock blocks, and state some known results concerning them.

Abacus combinatorics.

Let p be a natural number. It is often convenient for us to picture partitions on an abacus with p runners, following G. James ([46], pg. 78).

We thus label the runners on an abacus $0, \dots, p-1$, from left to right, and label its rows $0, 1, \dots$, from the top downwards. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition with m parts or fewer then we may represent λ on the abacus with m beads: for $i = 1, \dots, m$, write $\lambda_i + m - i = s + pt$, with $0 \leq s \leq p-1$, and place a bead on the s^{th} runner in the t^{th} row. For example,



is an abacus representation of the partition $(6, 4, 2^2, 1^2)$, when $p = 3$.

Sliding a bead one row upwards on its runner, into a vacant position, corresponds to removing a p -hook from the rim of the partition λ . Given an abacus representation of a partition, sliding all the beads up as far as possible produces an abacus representation of the p -core of that partition, a partition from which no further rim p -hooks can be removed. The pictured example $(6, 4, 2^2, 1^2)$ is therefore a 3-core. The p -core of a partition is independent of the way in which p -hooks have been removed. The p -weight of a partition is the number of p -hooks removed to obtain the p -core.

Fix an abacus representation of a partition λ , and for $i = 0, \dots, p-1$, let λ_1^i be the number of unoccupied positions on the i^{th} runner which occur above the lowest bead on that runner. Let λ_2^i be the number of unoccupied positions on the i^{th} runner which occur above the second lowest bead on that runner, etc. Then $\lambda^i = (\lambda_1^i, \lambda_2^i, \dots)$ is a partition, and the p -tuple $[\lambda^0, \dots, \lambda^{p-1}]$ is named the p -quotient of λ . The p -quotient depends on the

number of beads in the abacus representation of λ , but is well defined up to a cyclic permutation. The p -weight of λ is equal to the sum $|\lambda^0| + \dots + |\lambda^{p-1}|$.

The partitions with a given p -core τ and p -weight w may be parametrized by p -quotients:

Fix m so that any such partition has m parts or fewer. Representing the partitions on an abacus with m beads, there is a p -quotient for each one. We thus introduce a bijection between the set of partitions with p -core τ and weight w , and the set of p -tuples $[\sigma^0, \dots, \sigma^{p-1}]$ such that $|\sigma^0| + \dots + |\sigma^{p-1}| = w$. It is a convention of this article, frequently to label a partition by its p -quotient.

Whenever we represent partitions with a given p -core on an abacus, we assume that m is fixed as above.

Block parametrisation.

Let $A_{\mathcal{S}_q}(n) = \mathcal{S}_q(n, n)$, let $A_{\mathcal{H}_q}(n) = \mathcal{H}_q(\Sigma_n)$, and let $A_{G_q}(n) = kB_1$, a direct sum of the unipotent blocks of $kGL_n(q)$.

We give parametrizations for the blocks of $A_{\mathcal{X}}(n)$, for $\mathcal{X} \in \{\mathcal{S}_q, \mathcal{H}_q, G_q\}$. The parametrizations depend on an invariant p .

Definition 45 *If $\mathcal{X} \in \{\mathcal{S}_q, \mathcal{H}_q\}$, then $p = p(\mathcal{X})$ is defined to be the least natural number such that,*

$$1 + q + \dots + q^{p-1} = 0,$$

in k if such exists, and $p = \infty$, otherwise.

If $\mathcal{X} = G_q$, for some prime power q , then we insist that $l > 0$. In this case, $p = p(\mathcal{X})$ is defined to be the multiplicative order of q , modulo l .

Remark 46 Let $l > 0$. If q is a prime power such that $q \not\equiv 1$ modulo l , then $p(G_q) = p(\mathcal{S}_q)$.

However, if $q \equiv 1$ modulo l , then $p(G_q) = 1$, while $p(\mathcal{S}_q) = l$.

Theorem 47 *(R. Brauer, P. Fong, B. Srinivasan [31], 7A, G. James, R. Dipper, [25], 4.13, S. Donkin, A. Cox, [20]) The blocks of $A_{\mathcal{X}}(n)$ are in one-one correspondence with pairs (w, τ) , where $w \in \mathbb{N}_0$, and τ is a p -core of size $n - wp$.*

We write $b_{\tau, w}^{\mathcal{X}}$ for the block idempotent in $A_{\mathcal{X}}(n)$ corresponding to the pair (w, τ) , defined over \mathcal{O} . We write $k\mathbf{B}_{\tau, w}^{\mathcal{X}}$ for the corresponding block

algebra. We say that $b_{\tau,w}^{\mathcal{X}}$ (respectively $k\mathbf{B}_{\tau,w}^{\mathcal{X}}$) is a block of weight w , with core τ .

In this book, it will be convenient for us to work over an l -modular system (K, \mathcal{O}, k) . Thus, \mathcal{O} is a discrete valuation ring with field of fractions K of characteristic zero, maximal ideal \wp , and residue field $k = \mathcal{O}/\wp$ of characteristic l . For an \mathcal{O} -module M , we write $KM = K \otimes_{\mathcal{O}} M$, and $kM = k \otimes_{\mathcal{O}} M$.

Suppose $A_{\mathcal{X}}(n)$ is defined over \mathcal{O} . Abusing notation, we then write $b_{\tau,w}^{\mathcal{X}}$ for the lift over \mathcal{O} of the corresponding block idempotent defined over k . We write $R\mathbf{B}_{\tau,w}^{\mathcal{X}}$ for the corresponding block algebra over R , for $R \in \{K, \mathcal{O}, k\}$.

Remark 48 Here, and in the sequel, we write “block” in abbreviation of either “block idempotent”, or “block algebra”.

Theorem 49 (*R. Brauer, P. Fong, B. Srinivasan [31], 7A, G. James, R. Dipper, [25], 4.13, S. Donkin, A. Cox, [20]*) *Let τ be a partition of t , and a p -core. Let $n = pw + t$.*

(a) *A standard, costandard, or simple $\mathcal{S}_q(n, n)$ -module indexed by the partition λ lies in $k\mathbf{B}_{\tau,w}^{\mathcal{S}_q}$ if, and only if, λ has p -core τ , and p -weight w .*

(b) *A Specht, or simple $\mathcal{H}_q(\Sigma_n)$ -module indexed by the partition λ lies in $k\mathbf{B}_{\tau,w}^{\mathcal{H}_q}$ if, and only if, λ has p -core τ , and p -weight w .*

(c) *The irreducible character,*

$$\chi \left(((1)^{k_1}, (\sigma_2)^{k_2}, \dots, (\sigma_a)^{k_a}), (\lambda_1, \dots, \lambda_a) \right),$$

of $GL_n(q)$ lies in $k\mathbf{B}_{\tau,w}^{G_q}$ if, and only if, λ_1 has p -core τ , and the order of σ_i is a power of l , for $i \geq 2$. \square

We write $k\mathbf{B}_{\tau,w}^{\mathcal{S}}$ for the block $k\mathbf{B}_{\tau,w}^{\mathcal{S}_1}$ of the Schur algebra $\mathcal{S}(n, n)$. We write $k\mathbf{B}_{\tau,w}^{\Sigma}$ for the block $k\mathbf{B}_{\tau,w}^{\mathcal{H}_1}$ of the symmetric group Σ_n .

The following theorem is standard. Part (a) follows from the studies of P. Fong and B. Srinivasan [31]. Part (b) follows from work of R. Dipper and G. James [24].

Theorem 50 (a) *The blocks $k\mathbf{B}_{\tau,w}^{\Sigma}$ and $k\mathbf{B}_{\tau,w}^{G_q}$ have abelian defect groups if, and only if, $w < l$.*

(b) *If $w < l$, then a character $\chi \left(((1)^{k_1}, (\sigma_2)^{k_2}, \dots, (\sigma_a)^{k_a}), (\lambda_1, \dots, \lambda_a) \right)$ in $k\mathbf{B}_{\tau,w}^{G_q}$, is constrained by the condition that σ_i is of degree p , for $i \geq 2$. \square*

Derived equivalences, and Rock blocks.

Theorem 51 (*J. Chuang, R. Rouquier, [12]*) Let \mathcal{X} be an element of the set $\{\mathcal{S}_q, \mathcal{H}_q, G_q, \mathcal{S}, \Sigma\}$. Let τ, τ' be p -cores. Then the bounded derived categories $D^b(k\mathbf{B}_{\tau,w}^{\mathcal{X}})$, and $D^b(k\mathbf{B}_{\tau',w}^{\mathcal{X}})$, are equivalent. \square

Definition 52 Suppose p, w are fixed. A p -core ρ is said to be a Rouquier core if it has an abacus presentation, on which there are at least $w - 1$ more beads on runner i , than on runner $i - 1$, for $i = 1, \dots, p - 1$.

Example 53 Let $p = w = 3$. Then the partition $(6, 4, 2^2, 1^2)$ pictured on the abacus at the beginning of the chapter is a Rouquier core.

R. Rouquier conjectured the following structure theorem [64].

Theorem 54 (*J. Chuang, R. Kessar, [11]*) Let $w < l$, and let ρ be a Rouquier core. The block $k\mathbf{B}_{\rho,w}^{\Sigma}$ of a symmetric group is Morita equivalent to the wreath product $k\mathbf{B}_{\emptyset,1}^{\Sigma} \wr \Sigma_w$. \square

Let $\mathcal{X} \in \{\mathcal{S}_q, \mathcal{H}_q, G_q, \mathcal{S}, \Sigma\}$.

Definition 55 A Rock block is any block $b_{\rho,w}^{\mathcal{X}}$ (respectively $k\mathbf{B}_{\rho,w}^{\mathcal{X}}$), where ρ is a Rouquier core for p, w .

Remark 56

- (a) “RoCK block” is slang for “Rouquier, or Chuang-Kessar block”.
- (b) For fixed \mathcal{X}, p, w , all Rock blocks are Morita equivalent (cf. lemma 110).
- (c) Throughout this article, the letter ρ represents a Rouquier core. Other p -cores will be represented by different letters, such as τ .

The next couple of results were deduced from theorem 54.

Theorem 57 (*J. Chuang, K.M. Tan, [14]*) Let $w < l$. The Rock block $k\mathbf{B}_{\rho,w}^{\mathcal{S}}$ is Morita equivalent to $k\mathbf{B}_{\emptyset,1}^{\mathcal{S}} \wr \Sigma_w$. \square

For a natural number a , let Λ_w^a be the set of a -tuples $(\alpha^1, \dots, \alpha^a)$ of partitions α^i , such that $\sum_i |\alpha^i| = w$.

Let $c(\lambda; \mu, \nu)$ be the Littlewood-Richardson coefficient corresponding to partitions λ, μ, ν , taken to be zero whenever $|\lambda| \neq |\mu| + |\nu|$.

Theorem 58 (*J. Chuang, K.M. Tan, [14], H. Miyachi, [57]*) Let $w < l$. The decomposition matrix of the Rock block $k\mathbf{B}_{\rho,w}^S$ is equal to the decomposition matrix of $k\mathbf{B}_{\emptyset,1}^S \wr \Sigma_w$. It is given by,

$$d_{\underline{\lambda}\underline{\mu}} = \sum_{\alpha \in \Lambda_w^{p+1}, \beta \in \Lambda_w^p} \prod_{j=0}^{p-1} c(\lambda^j; \alpha^j, \beta^j) c(\mu^j; \beta^j, (\alpha^{j+1})'). \square$$

The following theorem was also observed, upon viewing the canonical basis for the basic representation of $\widehat{\mathfrak{sl}}_p$.

Theorem 59 (*J. Chuang and K.M. Tan, [13], B. Leclerc, and H. Miyachi, [53]*) Let $l = 0$, and let q be a p^{th} root of unity in k . The decomposition matrix of the Rock block $k\mathbf{B}_{\rho,w}^{S_q}$ is equal to the decomposition matrix of $k\mathbf{B}_{\emptyset,1}^{S_q} \wr \Sigma_w$. It is given by,

$$d_{\underline{\lambda}\underline{\mu}} = \sum_{\alpha \in \Lambda_w^{p+1}, \beta \in \Lambda_w^p} \prod_{j=0}^{p-1} c(\lambda^j; \alpha^j, \beta^j) c(\mu^j; \beta^j, (\alpha^{j+1})'). \square$$

Lemma 60 The p -regular partitions $[\lambda^0, \lambda^1, \dots, \lambda^{p-1}]$ with p -core ρ , are those for which λ^0 is empty. \square

Theorem 61 (*R. Paget, [59]*) Let ρ be a Rouquier core. Then

$$D^{[\emptyset, \lambda^1, \dots, \lambda^{p-1}]} \cong D^{[\emptyset, \lambda^{p-1}, \dots, \lambda^1]},$$

as $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ -modules. \square

Brauer correspondence for blocks of finite general linear groups.

If $a \in \mathcal{O}$, we denote the l -modular reduction of a by \bar{a} , an element of k . We insist that K is a splitting field for G .

Definition 62 (see [6]) For a finite group G , with an l -subgroup P , and an $\mathcal{O}G$ -module M , the Brauer homomorphism is defined to be the quotient map

$$Br_P^G : M^P \rightarrow M(P) = M^P / \left(\sum_{Q < P} Tr_Q^P(M^Q) + \wp M^P \right).$$

The Brauer quotient $M(P)$ is the quotient of P -fixedpoints of M by relative traces from proper subgroups, reduced modulo l ; $M(P)$ is a $kN_G(P)$ -module.

The Brauer homomorphism factors through the epimorphism $\mathcal{O}M^P \rightarrow kM^P$. Abusing notation, we write Br_P^G for the induced map from kM^P to $M(P)$.

In case that G acts on $M = \mathcal{O}G$ by conjugation, without doubt $\mathcal{O}G(P) = kC_G(P)$, and the quotient map

$$Br_P^G : (\mathcal{O}G)^P \rightarrow kC_G(P)$$

is an algebra homomorphism, the classical Brauer homomorphism with respect to P , given by the rule

$$Br_P^G(\sum_{g \in G} a_g g) = \sum_{g \in C_G(P)} \bar{a}_g g.$$

Theorem 63 (*L. Scott*) *Let M be a kG -permutation module. Then $M(P) \neq 0$ if, and only if, M has a direct summand with a vertex containing P . \square*

We present Brauer correspondence for unipotent blocks of finite general linear groups. Thus, let $l > 0$, and let q be a prime power, coprime to l .

Let $t = n - wp \geq 0$. Let

$$L_1 = \times^w GL_p(q) < GL_{wp}(q),$$

$$L_2 = GL_{wp}(q) \times GL_t(q) < GL_n(q),$$

be Levi subgroups. Let D be a Sylow l -subgroup of $GL_{wp}(q)$.

Let τ be a partition of t , and a p -core. Let $b_{\tau,w}^{G_q}$ (respectively $b_{\tau,0}^{G_q}$) be the unipotent block of $GL_n(q)$ (respectively $GL_t(q)$) with p -core τ .

The centralizer and normalizer of D in $GL_n(q)$ are contained in L_2 :

Lemma 64 (*[31], 1A, 3D, 3E*)

$$C_{GL_n(q)}(D) = C_{GL_{wp}(q)}(D) \times GL_t(q) \leq L_1 \times GL_t(q),$$

and,

$$N_{GL_n(q)}(D) = N_{GL_{wp}(q)}(D) \times GL_t(q) \leq (GL_p(q) \wr \Sigma_w) \times GL_t(q). \square$$

Theorem 65 (*M. Broué, [7], 3.5*) *The block $b_{\tau,w}^{G_q}$ has D as a defect group, and*

$$Br_D^{GL_n(q)}(b_{\tau,w}^{G_q}) = 1_{kC_{GL_{wp}(q)}D} \otimes b_{\tau,0}^{G_q},$$

where $Br_D^{GL_n(q)}$ is the classical Brauer morphism. \square

We record some information on these defect groups D of unipotent blocks of $GL_n(q)$, and on their centralizers and normalizers.

Let D_1 be a Sylow l -subgroup of $GL_p(q)$.

Lemma 66 ([31], 3D, 3E)

(a) $\times^w D_1$ is a Sylow l -subgroup of L_1 .

(b) D_1 is isomorphic to a cyclic group of order l^a , where

$$a = \max\{i \in \mathbb{Z}_{\geq 0} \mid l^i \text{ divides } q^p - 1\}.$$

(c) $D \cong \times^w D_1$ if, and only if, $w < l$. If $w \geq l$, then D is non-abelian.

(d) The normalizer of D in $GL_{wp}(q)$ is contained in the subgroup,

$$L_1 \rtimes \Sigma_w \cong GL_p(q) \wr \Sigma_w,$$

where Σ_w is the group of block permutation matrices, whose conjugation action on $\times^w GL_p(q)$ permutes the $GL_p(q)$'s. \square

Brauer correspondence for blocks of symmetric groups.

We now describe Brauer correspondence for blocks of symmetric groups. Therefore, for the rest of this section, $l = p > 0$.

Let $t = n - wp \geq 0$. Let

$$Y_1 = \times^w \Sigma_p < \Sigma_{wp},$$

$$Y_2 = \Sigma_{wp} \times \Sigma_t < \Sigma_n$$

be Young subgroups. Let D be a Sylow p -subgroup of Σ_{wp} .

Let τ be a partition of t , and a p -core. Let $b_{\tau,w}^\Sigma$ (respectively $b_{\tau,0}^\Sigma$) be the block of Σ_n (respectively Σ_t) with p -core τ .

Lemma 67 ([46], 4.1.19, 4.1.25)

$$C_{\Sigma_n}(D) = C_{\Sigma_{wp}}(D) \times \Sigma_t \leq Y_1 \times \Sigma_t,$$

and,

$$N_{\Sigma_n}(D) = N_{\Sigma_{wp}}(D) \times \Sigma_t \leq (\Sigma_p \wr \Sigma_w) \times \Sigma_t. \square$$

Theorem 68 (*M. Broué, L. Puig, [7], 2.3*) The block $b_{\tau,w}^\Sigma$ has D as a defect group, and

$$Br_{D'}^{\Sigma_n}(b_{\tau,w}^\Sigma) = 1_{kC_{\Sigma_{wp}}(D')} \otimes b_{\tau,0}^\Sigma,$$

for any p -subgroup D' of Σ_{wp} without fixed points in the set $\{1, 2, \dots, wp\}$.

We offer some information on these defect groups D , and their centralizers and normalizers.

Let D_1 be a Sylow p -subgroup of Σ_p . The following lemma is easily checked.

Lemma 69 (a) $\times^w D_1$ is a Sylow p -subgroup of Y_1

(b) $D_1 \cong C_p$, a cyclic group of order p .

(c) $D \cong \times^w D_1$ if, and only if, $w < p$. If $w \geq p$, then D is non-abelian.

(d) The normalizer of D in Σ_{wp} is contained in the subgroup

$$Y_1 \rtimes \Sigma_w \cong \Sigma_p \wr \Sigma_w,$$

where Σ_w is the group of block permutation matrices, whose conjugation action on $\times^w \Sigma_p$ permutes the Σ_p 's. \square

Morita equivalence for symmetric algebras.

Definition 70 An \mathcal{O} -order A , of finite rank, is said to be symmetric if there exists an \mathcal{O} -linear form $\phi : A \rightarrow \mathcal{O}$ such that,

(i) $\phi(aa') = \phi(a'a)$ for all $a, a' \in A$.

(ii) The map $\hat{\phi} : A \rightarrow \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$ defined by $\hat{\phi}(a)(a') = \phi(aa')$, for $a, a' \in A$, is an isomorphism of \mathcal{O} -modules.

M. Broué has given a sufficient condition for two \mathcal{O} -algebras to be Morita equivalent:

Theorem 71 ([8], 2.4) Let A and B be symmetric \mathcal{O} -orders of finite rank, and let M be an A - B bimodule which is projective of finite rank at the same time as a left A -module and as a right B -module. Suppose that the functor

$$KM \otimes_{KB} - : KB\text{-mod} \rightarrow KA\text{-mod},$$

is an equivalence of categories. Then the functor

$$M \otimes_B - : B\text{-mod} \rightarrow A\text{-mod},$$

is an equivalence of categories. \square

Chapter III

Rock blocks of finite general linear groups and Hecke algebras, when $w < l$.

We sketch a proof of the Morita equivalence of Rock blocks of finite general linear groups of abelian defect and their local blocks. Our proof of this theorem imitates Chuang and Kessar's proof of theorem 54. We subsequently deduce an analogous result for Rock blocks of Hecke algebras.

Rock blocks of finite general linear groups.

Let (K, \mathcal{O}, k) be an l -modular system. Let q be a prime power, coprime to l . Let $p = p(G_q)$ be the multiplicative order of q , modulo l .

Theorem 72 (*W. Turner [68], H. Miyachi [57]*) *Let $w < l$. The Rock block $\mathcal{O}b_{\rho,w}^{G_q}$ is Morita equivalent to the principal block, $\mathcal{O}b_{\emptyset,1}^{G_q} \wr \Sigma_w$, of $GL_p(q) \wr \Sigma_w$.*

For pedagogical purposes, we sketch the proof of this result. This will allow us to understand the similarities and differences as we approach Rock blocks of nonabelian defect ($w \geq l$), in later chapters. We include a proof of proposition 73, since this is relevant for the proof of theorem 80. Further details can be found elsewhere [68].

Let the Rouquier core $\rho = \rho(p, w)$ have size r . Let $v = wp + r$.

The principal block of $GL_p(q) \wr \Sigma_w$ is Morita equivalent to that block tensored with the defect 0 block $\mathcal{O}B_{\rho,0}^{G_q}$, a block of $(GL_p(q) \wr \Sigma_w) \times GL_r(q)$. We prove theorem 72 by showing that Green correspondence induces a Morita equivalence between this block of $(GL_p(q) \wr \Sigma_w) \times GL_r(q)$, and the block $\mathcal{O}B_{\rho,w}^{G_q}$ of $GL_v(q)$, when $w < l$.

Let $GL_v(q) = G = G_0 > G_1 > \dots > G_w = L$ be a sequence of Levi subgroups of $GL_v(q)$, where

$$G_i = GL_p(q)^i \times GL_{v-ip}(q).$$

Let

$$P_1 > \dots > P_w$$

be a sequence of parabolic subgroups of $G_0 > \dots > G_{w-1}$ with Levi subgroups

$$G_1 > \dots > G_w,$$

and unipotent complements

$$U_1 > \dots > U_w,$$

such that $P_i = U_i \rtimes G_i < G_{i-1}$.

Let $U_i^+ = \frac{1}{|U_i|} \sum_{u \in U_i} u$, a central idempotent in $\mathcal{O}U_i$.

Note that $|U_i|$ is a power of q^p , so equal to 1 (modulo l). Note also that G_i commutes with U_i^+ .

Let $a = b_{\emptyset,1}^{G_q}$ be the principal block idempotent of $GL_p(q)$. Let

$$b_i = a_1^{\otimes i} \otimes b_{\rho, w-i}^{G_q},$$

a block idempotent of G_i , for $1 \leq i \leq w$. We set $G = G_0$, $b = b_0$, and $f = b_w$.

Let Σ_w be the subgroup of permutation matrices of $GL_v(q)$ whose conjugation action on L permutes the factors of $GL_p(q)^i$.

We define N to be the semi-direct product of L and Σ_w , a subgroup of $GL_v(q)$ isomorphic to $(GL_p(q) \wr \Sigma_w) \times GL_r(q)$.

To prove theorem 72, we show that $\mathcal{O}Nf$ and $\mathcal{O}Gb$ are Morita equivalent, so long as $w < l$. It is not clear how to define the corresponding $\mathcal{O}Gb$ - $\mathcal{O}Nf$ -bimodule directly. However, we can describe the $\mathcal{O}Gb$ - $\mathcal{O}Lf$ -bimodule obtained by restriction.

Let $Y =_G Y_L = \mathcal{O}Gb_0 U_1^+ b_1 \dots U_w^+ b_w$, an $(\mathcal{O}Gb, \mathcal{O}Lf)$ -bimodule. The functor $Y \otimes_L -$ from L -mod to G -mod is

$$HCInd_{G_1 b_1}^{G_0 b_0} \dots HCInd_{G_w b_w}^{G_{w-1} b_{w-1}},$$

where $HCInd$ is Harish-Chandra induction.

Proposition 73 *The algebras $\mathcal{O}Nf$, and $End_G(Y)$ have the same \mathcal{O} -rank, equal to $w!dim_K(KLf)$.*

Proof:

The algebra $End_G(Y)$ is \mathcal{O} -free. It is therefore enough to compute the dimensions over K of $End_G(KY)$ and KNf . One of these is straightforward - ${}_N KNf$ is the induced module $Ind_L^N(KLf)$, so certainly has dimension $w!dim(KLf)$. The proposition will be proved when we have shown that $End_G(KY)$ has the same dimension.

We calculate,

$$_G KY =_G KY \otimes_L KLf = HCInd_{G_1 b_1}^{G_0 b_0} \dots HCInd_{G_w b_w}^{G_{w-1} b_{w-1}} (LKLf).$$

This is to be done by first computing $HCInd_{G_1 b_1}^{G_0 b_0} \dots HCInd_{G_w b_w}^{G_{w-1} b_{w-1}}(\psi)$, when ψ is an irreducible character of KLf , using the Littlewood-Richardson rule. The relevant combinatorics have already been described by Chuang and Kessar, and we record these below as lemma 74. Here, if $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \dots)$ are partitions, we write $\mu \subset \lambda$ exactly when $\mu_i \leq \lambda_i$ for $i = 1, 2, \dots$. An abacus is fixed so that the relevant p -quotients are well-defined.

Lemma 74 (*J. Chuang, R. Kessar [11], Lemma 4*) *Let λ be a partition with p -core ρ and weight $u \leq w$. Let $\mu \subset \lambda$ be a partition with p -core ρ and weight $u - 1$. Then there exists m with $0 \leq m \leq p - 1$ such that $\mu^l = \lambda^l$ for $l \neq m$ and $\mu^m \subset \lambda^m$ with $|\mu^m| = |\lambda^m| - 1$. Moreover the complement of the Young diagram of μ in that of λ is the Young diagram of the hook partition $(m + 1, 1^{p-m-1})$. \square*

In terms of character theory, by the Littlewood-Richardson rule, this means that Harish-Chandra induction from $K\mathbf{B}_{\emptyset,1}^{G_q} \otimes K\mathbf{B}_{\rho,u-1}^{G_q}$ to $K\mathbf{B}_{\rho,u}^{G_q}$ takes the character $\chi(1, (m + 1, 1^{p-m-1})) \otimes \chi(1, \mu)$ to the sum of $\chi(1, \lambda)$'s, such that λ is obtained from μ by moving a bead down the m^{th} runner.

Let us count the number of ways of sliding single beads down the e^{th} runner of a core j times, so that on the resulting runner the lowest bead has been lowered μ_1^e times, the second top bead has been lowered μ_2^e times, etc., so that $\mu_1^e \geq \mu_2^e \geq \dots$ and $\sum_i \mu_i^e = j$. It is equal to the number of ways of writing the numbers $1, \dots, j$ in the Young diagram of $(\mu_1^e, \mu_2^e, \dots)$ so that numbers increase across rows and down columns - that is, the degree of the character χ^{μ^e} of the symmetric group Σ_j (see [46], 7.2.7).

The characters in the block KLf are of the form,

$$\psi = \chi(s_1, \lambda_1) \otimes \dots \otimes \chi(s_w, \lambda_w) \otimes \chi(1, \rho),$$

where either $s_i = 1$, and λ_i is a p -core, or else $\lambda_i = (1)$, and $s_i = (\sigma_i)$ is given by a degree p element $\sigma_i \in \mathbb{F}_q^*$, whose order is a power of l , for $1 \leq i \leq w$.

A combinatorial description of the multiplicity of a given irreducible character of KGb in $KY \otimes_L \psi$ is now visible:

Suppose that s_1, \dots, s_{r_0} are all equal to 1, that $\lambda_i = (1)$ for $i \geq r_0 + 1$, that $\sigma_{r_0+1} \sim \dots \sim \sigma_{r_0+r_1} =: \theta_1$ are conjugate elements of \mathbb{F}_q^* , that $\sigma_{r_0+r_1+1} \sim$

$\dots \sim \sigma_{r_0+r_1+r_2} =: \theta_2$ are conjugate elements of \mathbb{F}_q^* , not conjugate to θ_1 , etc. etc. Also suppose that λ_i , for $i = 1, \dots, r_0$ is a p -hook, and that,

$$\lambda_1 = \dots = \lambda_{e_0} = (1^p),$$

$$\lambda_{e_0+1} = \dots = \lambda_{e_0+e_1} = (2, 1^{p-1}),$$

...

$$\lambda_{e_0+\dots+e_{p-2}+1} = \dots = \lambda_{e_0+\dots+e_{p-1}} = (p),$$

where $e_0 + \dots + e_{p-1} = r_0$. Then $KY \otimes_L \psi$ is equal to the character sum,

$$\sum (dim \chi^{\mu^0} \dots dim \chi^{\mu^{p-1}} \cdot dim \chi^{\nu^1} \cdot dim \chi^{\nu^2} \dots) \\ \chi((1, (\theta_1)^{r_1}, (\theta_2)^{r_2}, \dots), (\mu, \nu^1, \nu^2, \dots))$$

Here, the summation is over partitions $\mu = [\mu^0, \dots, \mu^{p-1}]$ of $|\rho| + r_0 p$ with core ρ , such that $(|\mu^0|, \dots, |\mu^{p-1}|) = (e_0, e_1, \dots, e_{p-1})$; over partitions ν^1 of r_1 ; over partitions ν^2 of r_2 , etc.

If, when we selected a character of L , we had permuted some of the (s_i, λ_i) 's (there are $(w!/e_0! \dots e_{p-1}! r_1! r_2! \dots)$ different arrangements), we would have seen the same character when we applied ${}_G KY \otimes_L -$. So the character of ${}_G KY$ is the sum of characters,

$$\sum_{|\mu^i|=e_i, |\nu^i|=r_i} \left[(w!/e_0! \dots e_{p-1}! r_1! r_2! \dots) \right. \\ \times dim \chi^{\mu^0} \dots dim \chi^{\mu^{p-1}} \cdot dim \chi^{\nu^1} \cdot dim \chi^{\nu^2} \dots \\ \times dim \chi(1, (1^p))^{e_0} \cdot dim \chi(1, (2, 1^{p-2}))^{e_1} \dots dim \chi(1, (p))^{e_{p-1}} \\ \times dim \chi(\rho_{X-1}) \times dim \chi(\theta_1, (1))^{r_1} \cdot dim \chi(\theta_2, (1))^{r_2} \dots \\ \left. \times \chi((1, (\theta_1)_1^r, (\theta_2)^{r_2}, \dots), ([\mu^0, \dots, \mu^{p-1}], \nu^1, \nu^2, \dots)) \right].$$

What is the dimension of the semisimple algebra $End_G(KY)$? Remembering that $\sum_{|\sigma|=m} |\chi^\sigma|^2 = m!$, it is,

$$\sum_{e_0+\dots+e_{p-1}+r_1+r_2+\dots=w} \left[(w!/e_0! \dots e_{p-1}! r_1! r_2! \dots)^2 \right. \\ \times e_0! \dots e_{p-1}! r_1! r_2! \dots \\ \times (dim \chi(1, (1^p)))^{2e_1} \cdot (dim \chi(1, (2, 1^{p-2})))^{2e_2} \dots (dim \chi(1, (p)))^{2e_p} \\ \left. \times (dim \chi(\rho_{X-1}))^2 \times (dim \chi(\theta_1, (1)))^{2r_1} \cdot (dim \chi(\theta_2, (1)))^{2r_2} \dots \right]$$

$$\begin{aligned}
&= w! \sum_{e_0 + \dots + e_{p-1} + r_1 + r_2 + \dots = w} \left[(w! / e_0! \dots e_{p-1}! r_1! r_2! \dots) \right. \\
&\times (\dim \chi(1, (1^p)))^{2e_0} \cdot (\dim \chi(1, (2, 1^{p-2})))^{2e_1} \dots (\dim \chi(1, (p)))^{2e_{p-1}} \\
&\times (\dim \chi(\rho_{X-1}))^2 \times (\dim \chi(\theta_1, (1)))^{2r_1} \cdot (\dim \chi(\theta_2, (1)))^{2r_2} \dots \left. \right] \\
&= w! \cdot \dim(KLf) \cdot \square
\end{aligned}$$

Let $D = D^1 \times \dots \times D^w$ be a Sylow l -subgroup of $GL_p(q)^1 \times \dots \times GL_p(q)^w$.

Lemma 75 *Let $w < l$.*

- (a) *D is a defect group of $\mathcal{O}G_i b_i$, for $i = 0, \dots, p-1$.*
- (b) *$Br_D^G(b_i) = 1_{kD} \otimes b_{\rho,0}^{G_q}$, and $Br_D^G(U_i^+) = 1$.*
- (c) *N stabilizes f , and as an $\mathcal{O}(N \times L)$ -module, $\mathcal{O}Nf$ is indecomposable with diagonal vertex ΔD . In particular, $\mathcal{O}Nf$ is a block of N .*
- (d) *$\mathcal{O}Gb$ and $\mathcal{O}Nf$ both have defect group D , and are Brauer correspondents. \square*

By the Brauer correspondence, the $\mathcal{O}(G \times G)$ -module $\mathcal{O}Gb$ and the $\mathcal{O}(N \times N)$ -module $\mathcal{O}Nf$ both have vertex ΔD and are Green correspondents. Let X be the Green correspondent of $\mathcal{O}Gb$ in $G \times N$, an indecomposable summand of $\text{Res}_{N \times N}^{G \times G}(\mathcal{O}Gb)$ with vertex ΔD . Because $\mathcal{O}Nf$ is a direct summand of $\text{Res}_{N \times N}^{G \times N}(X)$, we have $Xf \neq 0$, so $Xf = X$ and X is an $(\mathcal{O}Gb, \mathcal{O}Nf)$ -bimodule.

Theorem 72 is a consequence of the following:

Proposition 76 *Let $w < l$. Then ${}_G Y_L \cong {}_G X_L$, and $\mathcal{O}Nf \cong \text{End}_G(X) \cong \text{End}_G(Y)$ as algebras. The left $\mathcal{O}G$ -module ${}_G X$ is a progenerator for $\mathcal{O}Gb$. Hence $X \otimes_N -$ induces a Morita equivalence between $\mathcal{O}Nf$ and $\mathcal{O}Gb$. \square*

Remark 77 The correspondence between indecomposable modules of kNf and indecomposable modules of kGb given by Theorem 72 above is exactly Green correspondence between G and N . For if M is an indecomposable of kNf with vertex Q , the kG -module $X \otimes_{kN} M$ cannot have a smaller vertex

than Q , as then $M = X^* \otimes_{kG} X \otimes_{kN} M$ would have a smaller vertex than Q .

Remark 78 The proof of theorem 72 fails when $w \geq l$, because the indecomposable module ${}_GX_N$ does not restrict to an indecomposable module, ${}_GX_L$.

Rock blocks of Hecke algebras, when $w < l$.

Let us persist with the notation of the last section. Thus, (K, \mathcal{O}, k) is an l -modular system, and q is a prime power, coprime to l .

Lemma 79 Suppose A and B are \mathcal{O} -algebras, which are free \mathcal{O} -modules of finite rank, and suppose A and B are Morita equivalent via $F : A\text{-mod} \rightarrow B\text{-mod}$. Let e, f be idempotents in A, B . Then the following are equivalent:

- (i) eAe and fBf are Morita equivalent.
- (ii) $eS = 0$ if, and only if, $fF(S) = 0$, for all irreducible kA -modules S .

Theorem 72 squashes to Hecke algebras as follows:

Theorem 80 Let $w < l$. Let $p = p(G_q) = p(\mathcal{H}_q) > 1$. The Rock block $\mathcal{OB}_{\rho, w}^{\mathcal{H}_q}$ is Morita equivalent to $\mathcal{OB}_{\emptyset, 1}^{\mathcal{H}_q} \wr \Sigma_w$.

Proof:

Let $B_n(q)$ be a Borel subgroup of $GL_n(q)$ for $n = p, v$. Since $p > 1$, for $n = p, v$, the $\mathcal{O}GL_n(q)$ -module $\mathcal{O}[GL_n(q)/B_n(q)]$ is projective, having been induced from the projective trivial $\mathcal{O}B_n(q)$ -module.

We now construct a Morita bimodule from bimodules already available to us. Let $G = GL_v(q)$ be as in theorem 72. Let $H_w = \times^w GL_p(q)$. Let $G_w = H_w \rtimes \Sigma_w = GL_p(q) \wr \Sigma_w$. Let ${}_G Z_{G_w}$ be a bimodule inducing a Morita equivalence between $\mathcal{OB}_{\rho, w}^{G_q}$ and $\mathcal{OB}_{\emptyset, 1}^{G_q} \wr \Sigma_w$; such a bimodule exists by theorem 72. Let

$$\eta = \frac{1}{|B_p(q)|^w} \sum_{x \in B_p(q) \times \dots \times B_p(q)} x,$$

$$\xi = \frac{1}{|B_v(q)|} \sum_{b \in B_v(q)} b.$$

Then

$$\xi Z \eta$$

is an $\mathcal{H}_q(\Sigma_v)\text{-}\mathcal{H}_q(\Sigma_p) \wr \Sigma_w$ bimodule.

We need to show that under the Morita equivalence from $\mathcal{OB}_{\rho,w}^{G_q}$ to $\mathcal{OB}_{\emptyset,1}^{G_q} \wr \Sigma_w$; given by Z , simple kG -modules killed by η become simple kG_w -modules killed by ξ , and vice-versa. The truth of the theorem then follows from lemma 79.

The characters of G killed by ξ are the non-unipotent characters, by theorem 39. Under ${}_{H_w}Z^* \otimes_G -$, these become sums of characters $\chi^1 \otimes \dots \otimes \chi^w$, such that one of the χ^i 's is a non-unipotent character of $GL_p(q)$ (according to the proof of proposition 73). These are all killed by η . Conversely, by theorem 43, the characters of G_w killed by η are all composition factors of characters $Ind_{H_w}^{G_w}(\chi^1 \otimes \dots \otimes \chi^w)$, where one of the χ^i 's is a non-unipotent character of $GL_p(q)$. And ${}_GZ \otimes_{G_w} -$ sends this induced character to a sum of non-unipotent characters of G , all of which are killed by ξ .

The simples for G which vanish under multiplication by ξ are those $D(\lambda)$ indexed by p -singular partitions. These are simple composition factors of non-unipotent characters $\chi((1, (\sigma)^{|\nu|}), (\lambda_0, \nu))$, where ν is a non-empty partition, and σ is an l -element of \mathbb{F}_q^* of degree p ([10], Theorem 4.4d). But these characters correspond (under the Morita equivalence) to characters for G_w which are killed by η . Conversely, simple modules for G_w which are killed by η are composition factors of induced characters $Ind_{H_w}^{G_w}(\chi^1 \otimes \dots \otimes \chi^w)$, where one of the χ^i 's is a non-unipotent character of $GL_p(q)$. This becomes a character of G sent to zero by ξ on application of $Z \otimes_{G_w} -$. \square

Chapter IV

Rock blocks of symmetric groups, and the Brauer morphism.

The structure theorem of Chuang and Kessar states that Rock blocks of symmetric groups of abelian defect w are Morita equivalent to $k\mathbf{B}_{\emptyset,1}^{\Sigma} \wr \Sigma_w$ (theorem 54). The corresponding statement is false in nonabelian defect: the global and local blocks have different numbers of simple modules. Furthermore, the techniques developed to study Broué's abelian defect group conjecture [9] give little clue as to how to formulate an analogous result in nonabelian defect, let alone how to prove it.

Alperin's weight conjecture [1] suggests a deep uniformity in representation theory, which exists for all blocks of finite groups, and not only those of abelian defect. It therefore makes sense to search for an analogue of Chuang and Kessar's result, which is true for blocks arbitrary defect, and to develop techniques which may lead towards a proof. That is the dominant concern of this monograph. In chapter 8, we describe a conjecture, which predicts the structure of Rock blocks of symmetric groups of arbitrary defect. In other chapters, we give various numerical and structural results which point towards this conjecture, although none of them confirm it.

From a finite group representation theoretical perspective, our methods are somewhat eccentric, involving a peculiar application of the Brauer homomorphism, the theory of quasi-hereditary algebras, Ringel duality, cross-characteristic comparisons, quivers, Schur bialgebras, doubles, etc.. Standard tools, such as Green correspondence, appear to be all but useless in our situation. The resulting conjecture (conjecture 165) is simple in essence, and appears to be quite deep. It can be seen to be true in abelian defect, by comparison with Chuang and Kessar's structure theorem.

In this chapter, we introduce notation, for the study of Rock blocks of symmetric groups, of arbitrary defect. We show that in characteristic two, a Rock block is isomorphic to the group algebra of $\Sigma_2 \wr \Sigma_w$, once it has been cut by a certain idempotent (theorem 84). Our proof involves an unusual application of the Brauer homomorphism. We use this method to obtain a weaker result for Rock blocks of symmetric groups in arbitrary characteristic: we prove that the endomorphism ring of a certain l -permutation module M for $k\mathbf{B}_{\rho,w}^{\Sigma}$ is Morita equivalent to $k\Sigma_w$ (theorem 86).

Notation for Rock blocks of symmetric groups.

Throughout this chapter, we consider blocks of symmetric groups. Therefore, $l = p$.

Let (K, \mathcal{O}, k) be a p -modular system. Let w be a natural number. Let $\rho = \rho(p, w)$ be a Rouquier core, and a partition of r . Let $v = wp + r$.

Let $\Sigma_v = \text{Sym}\{1, 2, \dots, v\}$. Let $b_{\rho, w}^\Sigma$ be the Rock block of Σ_v . Let

$$L = \Sigma_p^1 \times \dots \times \Sigma_p^w \times \Sigma_r^0 \leq \Sigma_v,$$

where $\Sigma_p^i = \text{Sym}\{(i-1)p + 1, \dots, ip\}$, and $\Sigma_r^0 = \text{Sym}\{wp + 1, \dots, wp + r\}$. Let

$$f = b_{\emptyset, 1}^\Sigma \otimes \dots \otimes b_{\emptyset, 1}^\Sigma \otimes b_{\rho, 0}^\Sigma,$$

a block of L . Let

$$D = C_p^1 \times \dots \times C_p^w \leq L,$$

where C_p^i is the group of order p generated by the single element $((i-1)p + 1 \dots ip)$. Let

$$N = L \rtimes \Sigma_w \cong (\Sigma_p \wr \Sigma_w) \times \Sigma_r^0,$$

the normaliser of L in Σ_v . Since the centralizer $C_N(D) \leq L$, the idempotent f is also a block of N ([2] 15.1(5)).

Let e be an idempotent of $k\Sigma_v$, defined to the product $e_w \cdot e_{w-1} \dots e_0$ of block idempotents,

$$e_i = b_{\emptyset, 1}^\Sigma \otimes \dots \otimes b_{\emptyset, 1}^\Sigma \otimes b_{\rho, i}^\Sigma,$$

of $\Sigma_p^1 \times \dots \times \Sigma_p^{w-i} \times \Sigma_{ip+r}$.

Aping proposition 73, and its proof, we have,

Proposition 81 *The dimensions $\dim_k(ek\Sigma_v e)$ and $\dim_k(kNf)$ are both equal to $w! \cdot \dim_k(kLf)$. \square*

Conjecture 82 *There is an algebra isomorphism $ek\Sigma_v e \cong kNf$, for arbitrary w, p .*

Remark 83 The conjecture is true for $w < p$, by Chuang and Kessar [11]. The method of Chuang and Kessar fails for $w > p$, because the $k\Sigma_v$ - kNf bimodule $k\Sigma_v e$ fails to have diagonal vertex in this case, and consequently the Green correspondence cannot be used to give an abstract description of the bimodule. We prove the conjecture for $p = 2$ below, using the Brauer homomorphism.

R. Paget [60] has computed the projective summands of ${}_{\Sigma_v} k\Sigma_v e$, and shown that $ek\Sigma_v e$, and kNf have the same decomposition matrix. Her proof uses theorem 132, of this monograph.

Endomorphism rings.

Let P be the subgroup of D of order p generated by the element,

$$z = (1\dots p)(p+1\dots 2p)\dots((w-1)p+1\dots wp).$$

Let,

$$C = C_{\Sigma_v}(P) = D \rtimes \Sigma_w \times \Sigma_r^0 \leq N.$$

We consider the classical Brauer homomorphism - the surjective algebra homomorphism from $(k\Sigma_v)^P$ to kC which truncates elements of the group algebra at C . The images of block idempotents under the Brauer morphism are given by theorem 68. In particular,

$$Br_D^{\Sigma_v}(e) = Br_P^{\Sigma_v}(e) = Br_P^{\Sigma_v}(f) = 1_{kD} \otimes b_{\rho,0}^{\Sigma}.$$

Therefore,

$$Br_P^{\Sigma_v}((ek\Sigma_v e)^P) = Br_P^{\Sigma_v}(e).kC.Br_P^{\Sigma_v}(e) = kC.b_{\rho,0}^{\Sigma}.$$

If $p = 2$, then $N = C$, and $f = Br_P^{\Sigma_v}(f) = Br_P^N(f)$. In this case, we can prove conjecture 82:

Theorem 84 *If $p = 2$, then the Brauer homomorphism*

$$Br_P^{\Sigma_v} : (ek\Sigma_v e)^P \rightarrow kC$$

restricts to an isomorphism of algebras $(ek\Sigma_v e)^P \cong kNf$. Furthermore, $(ek\Sigma_v e)^P = ek\Sigma_v e$.

Proof:

From lemma 81, we know that $\dim_k(ek\Sigma_v e) = \dim_k(kNf)$. Also $N = C$, so that

$$\dim_k(ek\Sigma_v e)^P \leq \dim_k(ek\Sigma_v e) = \dim_k(kNf) = \dim_k(kCf).$$

But the Brauer homomorphism $Br_P^{\Sigma_v} : (ek\Sigma_v e)^P \rightarrow kCf$ is a surjection. So the dimensions above are all equal, and the theorem is proven. \square

Remark 85 Theorem 84 is false for $p > 2$, because kNf and $(ek\Sigma_v e)^P$ are both strictly larger than kCf .

If we cut down the module $k\Sigma_v e$, and the algebras that we are considering, it is possible to generalise the proof of theorem 84 to $p \geq 2$. More precisely, let $\zeta_w = \sum_{x \in \Sigma_p^1 \times \dots \times \Sigma_p^w} x$.

Let $M = \mathcal{O}\Sigma_v e\zeta_w$. Note that M is not a projective $\mathcal{O}\Sigma_v$ -module. Let $\mathcal{E} = \text{End}_{\Sigma_v}(M)$.

Since M is a p -permutation module, it is \mathcal{O} -free, and $\text{End}_{\Sigma_v}(RM) \cong R\mathcal{E}$, for $R \in \{K, \mathcal{O}, k\}$ ([51], 1.14.5).

To complete this chapter, we present the following theorem, as the consequence of a triad of lemmas. We expect the theorem to be true for $R = \mathcal{O}$, but we are unable to prove it, since our method involves the Brauer morphism, which takes values in a vector space over k .

Theorem 86 *Let $R \in \{K, k\}$. Then,*

$$R\mathcal{E} \cong R\Sigma_w \otimes R\mathbf{B}_{\rho,0}^{\Sigma}.$$

First, some notation. Let Θ_u be the set of partitions with core ρ of weight u obtained by moving beads only up the rightmost runner of the abacus representation of ρ . These correspond to partitions of u , via $[\emptyset, \dots, \emptyset, \nu] \leftrightarrow \nu$. In the usual notation for partitions, this correspondence is

$$(\rho_1 + p\nu_1, \rho_2 + p\nu_2, \dots, \rho_u + p\nu_u, \rho_{u+1}, \dots) \leftrightarrow (\nu_1, \nu_2, \dots, \nu_u).$$

Lemma 87 *The character of ${}_{\Sigma_v}KM$ is equal to*

$$\dim(\chi^{\rho}) \cdot \sum_{\nu} \dim(\chi^{\nu}) \cdot \chi^{[\emptyset, \dots, \emptyset, \nu]}.$$

Its endomorphism ring is isomorphic to $K\Sigma_w \otimes K\mathbf{B}_{\rho,0}^{\Sigma}$.

Proof:

We have ${}_{\Sigma_v}KM \cong K\Sigma_v e \otimes_L KLf\zeta_w$. This module is the KLf -module with character $\dim(\chi^{\rho}) \cdot (\chi^{(p)} \otimes \dots \otimes \chi^{(p)} \otimes \chi^{\rho})$, induced to $(\bigotimes^{w-1} K\Sigma_p) \otimes K\mathbf{B}_{\rho,1}^{\Sigma}$, then induced to $(\bigotimes^{w-2} K\Sigma_p) \otimes K\mathbf{B}_{\rho,2}^{\Sigma}$, etc., etc.. That its character is as stated follows from a symmetric group analogue of proposition 73, and its proof.

The endomorphism ring is isomorphic to $K\Sigma_w \otimes K\mathbf{B}_{\rho,0}^{\Sigma}$, since all algebras concerned are semisimple. \square

From now on, let us fix an isomorphism between the endomorphism ring of ${}_{\Sigma_v}KM$ and $K\Sigma_w \otimes K\mathbf{B}_{\rho,0}^{\Sigma}$, so that under the Morita equivalence, given by KM , between $K\mathbf{B}_{\rho,w}^{\Sigma}/\text{Ann}(KM)$ and $K\Sigma_w \otimes K\mathbf{B}_{\rho,0}^{\Sigma}$, the characters $\chi^{[\emptyset, \dots, \emptyset, \lambda]}$ and $\chi^{\lambda} \otimes \chi^{\rho}$ correspond.

Let $\zeta_D = \sum_{x \in D} x$. The sum ζ_D is the image of ζ_w under Br_P . Note that C commutes with ζ_D , and so kC acts on the right of $kC \cdot b_{\rho,0}^{\Sigma} \cdot \zeta_D$. Under this action, we have:

Lemma 88 *We have an isomorphism of algebras,*

$$kC.b_{\rho,0}^\Sigma / \text{Ann}(kC.b_{\rho,0}^\Sigma.\zeta_D) \cong k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma.$$

Proof:

It is true that $C \cong (D \rtimes \Sigma_w) \times \Sigma_r^0$, so that the top subgroup Σ_w commutes with ζ_D . Thus,

$$k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma \cong kC.b_{\rho,0}^\Sigma.\zeta_D$$

as vector spaces, via $x \mapsto x\zeta_D$. The annihilator $\text{Ann}_{kC}(kC.b_{\rho,0}^\Sigma.\zeta_D)$ is thus equal to $I(D).k\Sigma_w.b_{\rho,0}^\Sigma$ as a right kC -module, where Σ_w is the top group, and where $I(D)$ is the augmentation ideal of kD . Furthermore, the quotient of $kC.b_{\rho,0}^\Sigma$ by the annihilator acts on $k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma$ by the right regular action, and thus the quotient is actually isomorphic to $k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma$, as an algebra. \square

We may prove a version of lemma 87 over k , using Br_P :

Lemma 89 *We have an isomorphism of algebras,*

$$k\mathcal{E} \cong k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma.$$

The implicit action of $k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma$ on kM is,

$$m \circ (x \otimes y) = m. exye,$$

for $x \in k\Sigma_w$, $y \in k\mathbf{B}_{\rho,0}^\Sigma$.

Proof:

First observe that $k\mathbf{B}_{\rho,w}^\Sigma = ek\Sigma_v e \oplus *$ as L - L -bimodules, where $*$ has no summand with vertex containing ΔP , since,

$$ek\Sigma_v e(\Delta P) = Br_P(e).kC.Br_P(e) =$$

$$kC.b_{\rho,0}^\Sigma = k\mathbf{B}_{\rho,w}^\Sigma(\Delta P).$$

Likewise, kNf is a summand of $k\mathbf{B}_{\rho,w}^\Sigma$ as L - L -bimodules, where the complement has summands whose vertices do not contain ΔP . However, a vertex of the L - L -bimodule σkLf is $\Delta D^{(\sigma,1)} \geq \Delta P$ (for $\sigma \in \Sigma_w$). So, as an L - L -bimodule, kNf is the sum of summands of $k\mathbf{B}_{\rho,w}^\Sigma$ with vertices containing ΔP . Hence, $ek\Sigma_v e$ is a summand of kNf as an L - L -bimodule, and since their dimensions are equal by lemma 81, there are isomorphisms of L - L -bimodules,

$$ek\Sigma_v e \cong kNf \cong \bigoplus_{\sigma \in \Sigma_w} (\sigma kLf).$$

It follows that $\dim_k(ek\Sigma_v e\zeta_w) = w!. \dim_k(k\mathbf{B}_{\rho,0}^\Sigma)$.

Let A be the subalgebra of $(ek\Sigma_v e)^P$ generated by $ekCe$. Note that A acts on the right of $kM = k\Sigma_v e\zeta_w$ by multiplication, thus commuting with the left action of $k\Sigma_v$. In fact, A acts on the right of $(ek\Sigma_v e)^P \zeta_w$ by multiplication. Applying the Brauer homomorphism Br_P , we realise an action of $Br_P(A) = kC.b_{\rho,0}^\Sigma$ on the right of

$$Br_P((ek\Sigma_v e)^P \zeta_w) = Br_P((ek\Sigma_v e)^P).Br_P(\zeta_w) = kC.b_{\rho,0}^\Sigma \zeta_D.$$

by multiplication. Thus there is an algebra homomorphism, given by the composition,

$$A \rightarrow Br_P(A) \rightarrow kC.b_{\rho,0}^\Sigma / Ann(kC.b_{\rho,0}^\Sigma \cdot \zeta_D),$$

in the kernel of which lies $Ann_A((ek\Sigma_v e)^P \zeta_w)$. Lemma 88 implies the existence of a surjection,

$$A/Ann_A((ek\Sigma_v e)^P \zeta_w) \rightarrow k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma,$$

as well as the natural surjection,

$$A/Ann_A(k\Sigma_v e\zeta_w) \rightarrow A/Ann_A((ek\Sigma_v e)^P \zeta_w).$$

In addition, there is a sequence of natural injections,

$$\begin{aligned} A/Ann_A(k\Sigma_v e\zeta_w) &\rightarrow End_{\Sigma_v}(k\Sigma_v e\zeta_w) \rightarrow \\ &Hom_{\Sigma_v}(k\Sigma_v e, k\Sigma_v e\zeta_w) \rightarrow ek\Sigma_v e\zeta_w. \end{aligned}$$

But we have already agreed that $ek\Sigma_v e\zeta_w$ has dimension equal to,

$$w!. \dim_k k\mathbf{B}_{\rho,0}^\Sigma = \dim_k(k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma),$$

so in fact all of the above homomorphisms are isomorphisms. In particular,

$$End_{\Sigma_v}(kM) = End_{\Sigma_v}(k\Sigma_v e\zeta_w) \cong k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma. \square$$

Chapter V

Schur-Weyl duality inside Rock blocks of symmetric groups.

In this chapter, we strengthen theorem 84, and show that the Brauer homomorphism reveals Schur-Weyl duality between $\mathcal{S}(w, w)$, and $k\Sigma_w$, structurally inside the Rock block $k\mathbf{B}_{\rho, w}^\Sigma$ (theorem 90).

An alternative proof of the existence of a quotient of a symmetric group block, Morita equivalent to $\mathcal{S}(w, w)$ has been given by Cline, Parshall, and Scott, using Steinberg's tensor product theorem for the algebraic group GL_n ([19], 5.3). If the Reader inclines towards an understanding of blocks of finite groups, the proof given here should be interesting, because it is independent of algebraic group theory.

A Schur algebra quotient.

We assume the notation of chapter 4. Let $I = \text{Ann}_{R\mathbf{B}_{\rho, w}^\Sigma}(M)$, and let $B_{\rho, w}^\Sigma = \mathbf{B}_{\rho, w}^\Sigma/I$. Then I is \mathcal{O} -pure in $\mathbf{B}_{\rho, w}^\Sigma$, and so $RB_{\rho, w}^\Sigma = R\mathbf{B}_{\rho, w}^\Sigma/RI$, for $R \in \{K, \mathcal{O}, k\}$.

Theorem 90 *Let $R \in \{K, k\}$. Then $R\mathcal{E} = \text{End}_{\Sigma_w}(RM)$ is Morita equivalent to $R\Sigma_w$, and $RB_{\rho, w}^\Sigma$ is Morita equivalent to the Schur algebra $\mathcal{S}(w, w)$, defined over R .*

The $\mathcal{S}(w, w)$ - $R\Sigma_w$ -bimodule corresponding to the $RB_{\rho, w}^\Sigma$ - $R\mathcal{E}$ -bimodule RM is twisted tensor space, $E^{\otimes r\#}$.

Theorem 90 may be seen as a module theoretic interpretation of a theorem of Erdmann [30], which realises the decomposition matrix of the Schur algebra $\mathcal{S}(w, w)$ as a submatrix of the decomposition matrix of $k\mathbf{B}_{\rho, w}^\Sigma$. Indeed, we have the following interesting corollary:

Corollary 91 (*“Converse to Schur-Weyl duality”*) *Every block of polynomial $GL_n(k)$ -modules is Morita equivalent to a quotient of some symmetric group algebra, localised at an idempotent. \square*

Theorem 90 is clearly analogous to theorem 39 concerning general linear groups in non-describing characteristic. However, our methods hardly resemble those of Brundan, Dipper, and Kleshchev.

The proof of theorem 90 is the length of this chapter. We first find summands of kM as a right $k\Sigma_w \times \Sigma_r^0$ -module which are twisted Young modules for $k\Sigma_w$, tensored with the block $k\mathbf{B}_{\rho, 0}^\Sigma$. We then observe that all the indecomposable summands of kM are isomorphic to such twisted Young

modules. We finally show that the map $k\mathbf{B}_{\rho,w}^\Sigma \rightarrow \text{End}_{k\mathcal{E}}(kM)$ is surjective by an explicit calculation.

Let us state and prove some combinatorial preliminaries.

Lemma 92 *Let $\mu \notin \Theta_w$ be a partition of weight w with core ρ . Let $\theta \in \Theta_w$. Then $\mu \not\leq \theta$.*

Proof: In an abacus representation of μ obtained from ρ , at least one bead must be moved up a runner other than the rightmost runner. This means that μ_i is greater than ρ_i , for some $i > w$. Thus, $\mu_1 + \mu_2 + \dots + \mu_w$ is at most $\rho_1 + \rho_2 + \dots + \rho_w + (wp - 1)$. But $\theta_1 + \theta_2 + \dots + \theta_w = \rho_1 + \rho_2 + \dots + \rho_w + wp$. \square

Lemma 93 *Let $\theta \in \Theta_u$ be equal to $[\emptyset, \dots, \emptyset, \lambda]$, where λ is a partition of u . Let t be such that $u + t \leq w$. Then the character summand of*

$$\text{Ind}_{(\times^p \Sigma_t) \times \Sigma_{up+r}}^{\Sigma_{(t+u)p+r}} (\chi^{(1^t)} \otimes \dots \otimes \chi^{(1^t)} \otimes \chi^\theta)$$

obtained by removing all character summands indexed by partitions outside Θ_{u+t} is given by

$$\sum_{\nu} l_{\nu} \chi^{[\emptyset, \dots, \emptyset, \nu]},$$

where $\text{Ind}_{\Sigma_t \times \Sigma_u}^{\Sigma_{t+u}} (\chi^{(1^t)} \otimes \chi^\lambda) = \sum_{\nu} l_{\nu} \chi^{\nu}$.

Proof:

We have $\theta = (\rho_1 + p\lambda_1, \rho_2 + p\lambda_2, \dots, \rho_u + p\lambda_u, \rho_{u+1}, \dots)$. By the Littlewood-Richardson rule, the only characters obtained by inducing $\chi^{(1^t)} \otimes \chi^\theta$ to Σ_{t+up+r} are obtained by adding nodes onto t different rows of θ . Repeating this process p times, the only way of obtaining a character indexed by a partition in Θ_{t+u} is by adding nodes onto the same t rows p times, in such a way that the resulting partitions lie in Θ_{t+u} . The ways of doing this correspond exactly to the ways of adding a node to t different rows of λ , so that the resulting composition is a partition. These correspond exactly to the character summands of $\text{Ind}_{\Sigma_t \times \Sigma_u}^{\Sigma_{t+u}} (\chi^{(1^t)} \otimes \chi^\lambda)$, by the Littlewood-Richardson rule. \square

Permutation modules for Σ_v .

Suppose that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ is a partition of w . Let $j_\lambda^1 = 0$, and let j_λ^i be the sum $\sum_{r=1}^{i-1} \lambda_r$ ($i = 2, 3, \dots, N$). Let J_λ^i ($i = 1, 2, \dots, N$) be the subgroup,

$$\text{Sym}\{pj_\lambda^i + 1, pj_\lambda^i + p + 1, \dots, pj_\lambda^i + p(\lambda_i - 1) + 1\}$$

$$\begin{aligned} & \times \text{Sym}\{pj_\lambda^i + 2, pj_\lambda^i + p + 2, \dots, pj_\lambda^i + p(\lambda_i - 1) + 2\} \\ & \times \dots \times \text{Sym}\{pj_\lambda^i + p, pj_\lambda^i + p + p, \dots, pj_\lambda^i + p\lambda_i\} \end{aligned}$$

of Σ_v , a product of p symmetric groups, each of which is isomorphic to S_{λ_i} . Note that J_λ^i has support $\{m \in \mathbb{Z} | pj_\lambda^i + 1 \leq m \leq pj_\lambda^{i+1}\}$. Let

$$J_\lambda = J_\lambda^1 \times J_\lambda^2 \times \dots \times J_\lambda^N,$$

a subgroup of Σ_v isomorphic to $\times^p \Sigma_\lambda$, a direct copy of p copies of the Young subgroup Σ_λ of Σ_w .

Recall that Σ_r^0 is defined to be $\text{Sym}\{wp + 1, \dots, wp + r\}$. We define

$$\Sigma_{r+pj_\lambda^i}^0 = \text{Sym}\{1, 2, 3, \dots, pj_\lambda^i, wp + 1, wp + 2, \dots, wp + r\}.$$

This is the symmetric group whose support is equal to the support of the direct product $J_\lambda^1 \times \dots \times J_\lambda^{i-1} \times \Sigma_r^0$.

Let $\xi_\lambda^i = \sum_{y \in J_\lambda^i} y$, for $i = 1, 2, \dots, N$. Let

$$\xi_\lambda = (\xi_\lambda^1 \xi_\lambda^2 \dots \xi_\lambda^N) b_{\rho,0}^\Sigma \cdot b_{\rho,\lambda_1}^\Sigma \cdot b_{\rho,\lambda_1+\lambda_2}^\Sigma \dots b_{\rho,w}^\Sigma,$$

an element of $\mathcal{O}\Sigma_v$. In this formula, we take $b_{\rho,j_\lambda^i}^\Sigma$ to be an element of $\mathcal{O}\Sigma_{r+pj_\lambda^i}^0$. Let $\eta_\lambda^i = \sum_{y \in J_\lambda^i} \text{sgn}(y) \cdot y$, for $i = 1, 2, \dots, N$. Let

$$\eta_\lambda = (\eta_\lambda^1 \eta_\lambda^2 \dots \eta_\lambda^N) b_{\rho,0}^\Sigma \cdot b_{\rho,\lambda_1}^\Sigma \cdot b_{\rho,\lambda_1+\lambda_2}^\Sigma \dots b_{\rho,w}^\Sigma,$$

an element of $\mathcal{O}\Sigma_v$.

Consider $\mathcal{O}\Sigma_v \xi_\lambda$. This $\mathcal{O}\Sigma_v$ -module may be constructed as follows:

Take the projective module $\mathcal{O}\Sigma_r$ -module $\mathcal{O}\mathbf{B}_{\rho,0}^\Sigma$. Tensor this module with p copies of the Young module $Y^{(\lambda_1)}$, each of which is isomorphic to the trivial module for $\mathcal{O}\Sigma_{\lambda_1}$. Cut the resulting p -permutation module off at the block of $\Sigma_{r+p\lambda_1}$ with core ρ . Now tensor this module with p copies of $Y^{(\lambda_2)}$ and cut off at the block of $\Sigma_{r+p\lambda_1+p\lambda_2}$ with core ρ . Repeat this process until a p -permutation module for $\mathcal{O}\Sigma_v$ is obtained, in the block with core ρ .

Why can $\mathcal{O}\Sigma_v \xi_\lambda$ be constructed in this way? Because ξ_λ^i generates the trivial module for J_λ^i , which is isomorphic to p copies of the symmetric group Σ_λ^i . And because the idempotent $b_{\rho,j_\lambda^i}^\Sigma$ commutes with the subgroup $\Sigma_{r+pj_\lambda^i}^0$.

We have a similar construction of $\mathcal{O}\Sigma_v \eta_\lambda$. This time, rather than tensoring with p copies of the trivial module each time before inducing, you should tensor with p copies of the alternating module.

When searching for summands of the right $k\Sigma_w \times \Sigma_r^0$ -module kM , we will be interested in the projective part of $\mathcal{O}\Sigma_v\eta_\lambda = (\mathcal{O}\Sigma_v\xi_\lambda)^\#$.

Pursuing this, let us first note that $\mathcal{O}\Sigma_v\xi_\lambda$ is a p -permutation module. This is because a direct summand of a module induced from a trivial module tensored with a sum of Young modules, from a Young subgroup; the module $\mathcal{O}\mathbf{B}_{\rho,0}^\Sigma$ for $\mathcal{O}\Sigma_r^0$ is isomorphic to a sum of Young modules, since that block has defect zero.

Hence it is a direct sum of Young modules. It is known that a Young module Y^μ is projective if and only if μ is p -restricted. Also, a Young module Y^μ has a Specht filtration, and the Specht subquotients S^γ occuring satisfy $\gamma \supseteq \mu$.

So any Young module summand of $\mathcal{O}\Sigma_v\xi_\lambda$ which is not projective only has Specht quotients S^γ where $\gamma \supseteq \mu$ for some p -restricted partition μ .

Now, $\mathcal{O}\Sigma_v\eta_\lambda = (\mathcal{O}\Sigma_v\xi_\lambda)^\# \cong \mathcal{O}\Sigma_v\xi_\lambda \otimes \text{sgn}$, where sgn is the signature representation. And we know that $S^\gamma \otimes \text{sgn} \cong (S^{\gamma'})^*$, by Remark 26. Furthermore, tensoring with sgn preserves indecomposability and takes projectives to projectives and non-projectives to non-projectives.

Thanks to the above discussion, we know the following fact:

Lemma 94 *Let $\mathcal{O}\Sigma_v\eta_\lambda = U \oplus V$, where U is projective and V has no projective summand. Then the character of KV has irreducible constituents χ^δ marked by $\delta \trianglelefteq \beta$ where $\beta = \beta(\delta)$ is p -singular. \square*

Let K_λ^i ($i = 1, 2, \dots, N$) be the diagonal subgroup of the direct product J_λ^i (here we fix isomorphisms between the factors of J_λ^i which preserve the ordering on $\{1, 2, \dots, wp\}$), a group isomorphic to Σ_{λ_i} . Let

$$K_\lambda = K_\lambda^1 \times K_\lambda^2 \times \dots \times K_\lambda^N,$$

a subgroup of Σ_v isomorphic to a Young subgroup Σ_λ of Σ_w .

Note that $N \cong L \rtimes K_{(w)}$, and $K_\lambda < K_{(w)}$ normalises D . The following lemma is straightforward:

Lemma 95 (a) *The direct product $J_\lambda^i \cong \times_p \Sigma_{\lambda_i}$ is normalised by P . The generator z of P acts on J_λ^i by circulating the p direct factors of J_λ^i .*

(b) *The direct product $J_\lambda \cong \times_p \Sigma_\lambda$ is normalised by P . The generator z of P acts on J_λ by circulating the p direct factors of J_λ .*

(c) *The diagonal subgroup K_λ^i of J_λ^i is equal to $C_{\Sigma_v}(P) \cap J_\lambda^i$. The diagonal subgroup K_λ of J_λ is equal to $C_{\Sigma_v}(P) \cap J_\lambda$.*

(d) We have $Br_P(\xi_\lambda^1 \xi_\lambda^2 \dots \xi_\lambda^N) = \sum_{K_\lambda} y$, and $Br_P(\eta_\lambda^1 \eta_\lambda^2 \dots \eta_\lambda^N) = \sum_{K_\lambda} sgn(y).y \square$

Summands of $M_{\mathcal{E}}$ as Young modules.

Recall from the proof of theorem 86 that the algebra A acts on the right of the module kM and its image is the endomorphism ring $k\mathcal{E}$. The algebra A commutes with the conjugation action of P , and thus preserves $(kM)^P$ in its action. Hence, $k\mathcal{E} \cong k\Sigma_w \otimes k\mathbf{B}_{\rho,0}^\Sigma$ preserves $(kM)^P$ in its action.

Lemma 96 (a) We have $\eta_\lambda \in (\mathcal{O}\Sigma_v)^P$, and,

$$Br_P(\eta_\lambda) = \left(\sum_{K_\lambda} sgn(y).y \right).b_{\rho,0}^\Sigma.$$

(b) $Br_P(\eta_\lambda(kM)^P) \cong Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes k\mathbf{B}_{\rho,0}^\Sigma)$, as a right $\Sigma_w \times \Sigma_r^0$ -module.

Proof:

Note that P commutes with $b_{\rho,j_\lambda^i}^\Sigma$ for every i , and $Br_P(b_{\rho,j_\lambda^i}^\Sigma) = b_{\rho,0}^\Sigma$. Since the Brauer map is an algebra homomorphism, we compute the image under Br_P of the product η_λ to be

$$Br_P(\eta) = \left(\sum_{K_\lambda} sgn(y).y \right).b_{\rho,0}^\Sigma.$$

This completes the proof of (a). Thus,

$$\begin{aligned} & Br_P(\eta_\lambda.(kM)^P) \\ &= Br_P(\eta_\lambda(k\Sigma_v)^P e\zeta_w) \\ &= \left(\sum_{y \in K_\lambda} sgn(y)y \right).kC.b_{\rho,0}^\Sigma.\zeta_D \\ &= \zeta_D.\left(\sum_{y \in K_\lambda} sgn(y)y \right).kC.b_{\rho,0}^\Sigma, \end{aligned}$$

which is isomorphic to $Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes k\mathbf{B}_{\rho,0}^\Sigma)$ as a right $k\Sigma_w \times \Sigma_r^0$ -module. \square

Proposition 97 Let λ be a partition of w . There is an idempotent f_λ in $\mathcal{O}\Sigma_v$ such that,

(a) $f_\lambda kM$ is isomorphic to $Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes K\mathbf{B}_{\rho,0}^\Sigma)$, as a right $K\Sigma_w \times \Sigma_r^0$ -module.

(b) $f_\lambda kM$ is isomorphic to $Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes k\mathbf{B}_{\rho,0}^\Sigma)$, as a right $k\Sigma_w \times \Sigma_r^0$ -module.

(c) $f_\lambda kM = \eta_\lambda kM = \eta_\lambda(kM)^P \cong Br_P(\eta_\lambda(kM)^P)$.

Proof:

By lemma 94, we can write the right $|cal\mathcal{O}\Sigma_v$ -module, $\eta_\lambda\mathcal{O}\Sigma_v = U \oplus V$, where U is projective and V has no projective summand. And that the character of KV only has constituents labelled by $\delta \leq \beta$ where $\beta = \beta(\delta)$ is p -singular. Since $\eta_\lambda\mathcal{O}\Sigma_v$ is a quotient of $\mathcal{O}\Sigma_v$, the submodule U may be generated by an idempotent. Let f_λ be such an idempotent, so that $U = f_\lambda\mathcal{O}\Sigma_v$. We prove (a) and (b) for this idempotent.

Note that the character of the left module $K\Sigma_v f_\lambda$ is the same as the character of the right module $f_\lambda K\Sigma_v$, since all symmetric group characters are real, and therefore self-dual. Similarly, the character of the left module $K\Sigma_v \eta_\lambda$ is the same as the character of the right module $\eta_\lambda K\Sigma_v$,

First we show that for $\gamma \in \Theta_w$, we have

$$(\chi^\gamma, \text{char}(K\Sigma_v \eta_\lambda)) = (\chi^\gamma, \text{char}(K\Sigma_v f_\lambda)),$$

where $\text{char}(X)$ denotes the character of a $K\Sigma_v$ -module X .

To see this, only observe that for $\gamma \in \Theta_w$, the formula $(\chi^\gamma, \text{char}(KV)) = 0$ holds, for the complement V of U defined above. This is a consequence of lemma 92, for Θ_w contains no p -singular partitions.

As our second task, we show that $\eta_\lambda KM = f_\lambda KM$, Indeed,

$$\eta_\lambda KM \cong \text{Hom}_{\Sigma_v}(K\Sigma_v \eta_\lambda, KM),$$

(note that the algebra $K\Sigma_v$ is semisimple). And by lemma 87, $\text{char}(KM)$ only has constituents in Θ_w . As has already been established, the part of $K\Sigma_v \eta_\lambda$ in Θ_w is the same as the part of $K\Sigma_v f_\lambda$ in Θ_w , so

$$\text{Hom}_{\Sigma_v}(K\Sigma_v \eta_\lambda, KM) \cong$$

$$\text{Hom}_{\Sigma_v}(K\Sigma_v f_\lambda, KM) \cong f_\lambda KM.$$

So $f_\lambda KM \subseteq \eta_\lambda KM$, and the two spaces have the same dimension. Therefore $f_\lambda KM = \eta_\lambda KM$, as required.

As our third task, we show that $\eta_\lambda kM = f_\lambda kM$. Note that the embedding $f_\lambda M \hookrightarrow \eta_\lambda M$ splits via left multiplication by f_λ , and therefore $f_\lambda M$ is \mathcal{O} -pure in $\eta_\lambda M$. However, $f_\lambda M$ and $\eta_\lambda M$ have the same dimension over K , and therefore over \mathcal{O} . Therefore $f_\lambda M = \eta_\lambda M$, and $\eta_\lambda kM = f_\lambda kM$.

As a fourth task, we should convince ourselves that the character component of $K\Sigma_v \eta_\lambda$ which corresponds to Θ_w is equal to

$$(*) \quad \dim(\chi^\rho) \cdot \sum_{\nu \vdash w} l_\nu \chi^{[\emptyset, \dots, \emptyset, \nu]},$$

where l_ν is defined by $Ind_{\Sigma_\lambda}^{\Sigma_w}(sgn \otimes \dots \otimes sgn) = \sum_\nu l_\nu \chi^\nu$.

From this formula, part (a) of this proposition follows directly - recall that $-\otimes_{\Sigma_v} KM$ matches the character $\chi^{[\emptyset, \dots, \emptyset, \lambda]}$ of $K\mathbf{B}_{\rho, w}^\Sigma$ with the character $\chi^\lambda \otimes \chi^\rho$ of $K\Sigma_w \otimes K\mathbf{B}_{\rho, 0}^\Sigma$.

To see (*), consider our construction of $\mathcal{O}_{\Sigma_v} \eta_\lambda$ as a module, tensored, induced and cut, tensored, induced and cut,.... At the same time meditate upon lemma 93. The formula is then visible, inductively.

Our final task is to prove part (b) of the proposition. The Brauer map gives (by lemma 88) a surjection

$$\eta_\lambda(kM)^P \rightarrow Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes k\mathbf{B}_{\rho, 0}^\Sigma).$$

In addition, there is an injection

$$\eta_\lambda(kM)^P \rightarrow \eta_\lambda kM = f_\lambda kM.$$

But part (a) of the proposition shows that the dimension of $f_\lambda kM$ is equal to the dimension of $Ind_{\Sigma_\lambda \times \Sigma_r^0}^{\Sigma_w \times \Sigma_r^0}(sgn \otimes k\mathbf{B}_{\rho, w}^\Sigma)$, so these two maps must be isomorphisms. This completes the proof of (b) and (c). \square

Lemma 98 *The right $k\Sigma_w \times \Sigma_r^0$ -module kM is isomorphic to a sum of summands of $\bigoplus_\lambda f_\lambda kM$.*

Proof:

Let us write \mathcal{O}_{Σ_v} as a direct sum of projective indecomposable modules. Let j be an idempotent, such that $_{\Sigma_v} \mathcal{O}_{\Sigma_v} j$ is the sum of projective indecomposables in this decomposition with simple tops $\{D^\lambda | \lambda \in \Theta_w\}$.

Note that $\mathcal{O}_{\Sigma_v} f_\lambda$ has the projective cover of $D^{[\emptyset, \dots, \emptyset, \lambda]}$ as a summand - because its character is $\chi^{[\emptyset, \dots, \emptyset, \lambda]} + (\text{a sum of } \chi^\mu \text{'s, } \mu \not\leq [\emptyset, \dots, \emptyset, \lambda])$, by the formula (*) and lemma 92. Thus, $\bigoplus_{\lambda \vdash w} f_\lambda M$ has every summand of jM as a summand.

But $jM = M$, since

$$i\mathcal{O}_{\Sigma_v} e\Sigma_w = Hom_{\Sigma_v}(\mathcal{O}_{\Sigma_v} i, M) = 0$$

for any projective indecomposable module $\mathcal{O}_{\Sigma_v} i$ with simple top outside $\{D^\lambda | \lambda \in \Theta_w\}$ (recall KM has character summands corresponding to elements of Θ_w , and $K\Sigma_v i$ has character summands corresponding to partitions outside Θ_w by lemma 92).

The lemma holds ! \square

Let G be a finite group, with subgroups H and K . Let $\zeta_H = \sum_{h \in H} h$ and let $\zeta_K = \sum_{k \in K} k$ be corresponding sums in the group algebra kG .

Lemma 99 (a) As kG -modules, $kG.\zeta_H \cong k[G/H]$ and $kG.\zeta_K \cong k[G/K]$.

(b) As vector spaces, $\text{Hom}_G(kG\zeta_H, kG\zeta_K) \cong k[H \backslash G/K]$

(c) Any element of $\text{Hom}_G(kG.\zeta_H, kG.\zeta_K)$ can be written as right multiplication by some element of kG .

Proof:

The map $g\zeta_H \mapsto gH$, for $g \in G$ defines a kG -module isomorphism $kG\zeta_H \cong k[G/H]$. Furthermore,

$$k[H \backslash G/K] \cong \text{Hom}_G(k[G/H], k[G/K]),$$

$$HgK \mapsto (H \mapsto \sum_{x \in HgK/K} xK).$$

In other words, $\text{Hom}_G(kG\zeta_H, kG\zeta_K)$ has a basis T_s indexed by elements $s \in H \backslash G/K$, where

$$T_{HgK} : \zeta_H \mapsto \sum_{x \in HgK} x = \zeta_H \cdot g \cdot \sum_{y \in T} y,$$

and T is a set of representatives for $(g^{-1}Hg \cap K) \backslash K$. So T_{HgK} can be defined as right multiplication by $\sum_{y \in T} y$. \square

Lemma 100 The natural map $f_\lambda k\Sigma_v f_\mu \rightarrow \text{Hom}_{k\mathcal{E}}(f_\mu kM, f_\lambda kM)$ is surjective.

Proof:

Since $f_\lambda kM = \eta_\lambda kM$, left multiplication by $x \in k\Sigma_v$ on $f_\lambda kM \subseteq k\Sigma_v$ is equivalent to left multiplication by $xf_\lambda \in k\Sigma_v$. Similarly, if $x.y \in \eta_\mu kM = f_\mu kM$, then $f_\mu x.y = x.y$. To prove the lemma then, it is sufficient to show that every element of $\text{Hom}_{\mathcal{E}}(\eta_\lambda kM, \eta_\mu kM)$ is given (as left multiplication) by an element of $k\Sigma_v$.

Recall from lemma 97 and the proof of lemma 96 that,

$$\begin{aligned} \eta_\lambda kM &= \eta_\lambda (kM)^P \cong \text{Br}_P(\eta_\lambda (kM)^P) \\ &= \zeta_D \cdot \left(\sum_{y \in K_\lambda} \text{sgn}(y)y \right) \cdot kC \cdot b_{\rho,0}^\Sigma \\ &\cong \left(\sum_{y \in K_\lambda} \text{sgn}(y)y \right) kK_{(w)} \otimes k\mathbf{B}_{\rho,0}^\Sigma. \end{aligned}$$

Thanks to lemma 99, any homomorphism between,

$$\left(\sum_{y \in K_\mu} \text{sgn}(y).y \right) kK_{(w)}, \text{ and}$$

$$\left(\sum_{y \in K_\lambda} \text{sgn}(y).y \right) kK_{(w)},$$

is given as left multiplication by an element of $kK_{(w)}$. See this by applying the automorphism $\#$. Thus, any homomorphism between,

$$\eta_\mu kM \cong \left(\sum_{y \in K_\mu} \text{sgn}(y).y \right) kK_{(w)} \otimes k\mathbf{B}_{\rho,0}^\Sigma, \text{ and}$$

$$\eta_\lambda kM \cong \left(\sum_{y \in K_\lambda} \text{sgn}(y).y \right) kK_{(w)} \otimes k\mathbf{B}_{\rho,0}^\Sigma,$$

is given as left multiplication by an element z of $kK_{(w)} \otimes k\mathbf{B}_{\rho,0}^\Sigma$.

To complete the proof of the lemma, we need to show we can choose a preimage of $z \in kK_{(w)}$ under Br_P which sends $\eta_\mu(kM)^P$ to $\eta_\lambda(kM)^P$. There is a diagonal embedding $\Delta : \Sigma_w \hookrightarrow \times^p \Sigma_w \cong J_{(w)}$, whose image in $J_{(w)}$ is $K_{(w)}$. The group P acts on $J_{(w)}$ by rotating the p copies of Σ_w . Thinking of z as an element of $k\Sigma_w$, we can consider the element $t = z^{\otimes p} \in kJ_{(w)}$. Note that t commutes with P . Writing $z = \sum_{g \in \Sigma_w} a_g g$ as a linear combination of group elements g , we compute

$$Br_P(t) = \sum (a_g)^p \Delta(g).$$

Since \mathbb{F}_p is a splitting field, we may take $k = \mathbb{F}_p$, so that $(a_g)^p = a_g$, for all a_g , and thus

$$Br_P(t) = \sum a_g \Delta(g) = z \in kK_{(w)}.$$

Multiplying on the left by $i.t$, where i is the idempotent factor of η_μ defines a map $\eta_\mu(kM)^P \rightarrow \eta_\lambda(kM)^P$. Computing the effect of this map via the Brauer morphism, we find it corresponds to multiplication by z inside $K_{(w)}$, as required. \square

Corollary 101 *The natural algebra homomorphism from $k\mathbf{B}_{\rho,w}^\Sigma$ to the endomorphism ring $End_{k\mathcal{E}}(kM)$ is a surjection.*

Proof:

The algebra $k\mathcal{E}$ is defined to be the endomorphism ring $End_{\Sigma_v}(kM)$. So $k\Sigma_v$ maps to $End_{k\mathcal{E}}(kM)$.

By proposition 97 (b), and lemma 98, and Schur-Weyl duality, $End_{k\mathcal{E}}(kM)$ is Morita equivalent to the Schur algebra, since the indecomposable summands of tensor space as a symmetric group module are precisely the twisted

Young modules. The collection of isomorphism classes of irreducible $End_{k\mathcal{E}}(kM)$ -modules is therefore in bijection with the collection of partitions of w .

The algebra $k\mathbf{B}_{\rho,w}^\Sigma/kI$ is Morita equivalent to a quotient of the Schur algebra $\mathcal{S}(v, v)$, with isomorphism classes of irreducible modules in bijection with Θ_w , which is in bijection with the collection of partitions of w . The algebras $End_{k\mathcal{E}}(kM)$ and $k\mathbf{B}_{\rho,w}^\Sigma/kI$ therefore have the same number of isomorphism classes of irreducible modules.

Thanks to lemma 98 and lemma 100, for any primitive idempotents $i, j \in k\mathbf{B}_{\rho,w}^\Sigma/kI$, the natural map

$$i(k\mathbf{B}_{\rho,w}^\Sigma/kI)j \rightarrow Hom_{\mathcal{E}}(jkM, ikM)$$

is surjective. Since $k\mathcal{E}$ and $k\mathbf{B}_{\rho,w}^\Sigma/kI$ have the same number of isomorphism classes of irreducible modules, the map from $k\Sigma_v$ to $End_{k\mathcal{E}}(kM)$ is surjective. \square

Remark 102 In the light of theorem 97 and lemma 98, theorem 90 is proven. \square

Remarks and questions.

Recall that the set of irreducible characters for the Schur algebra $K\mathcal{S}(w, w)$ may be parametrized $\{\chi(\lambda) | \lambda \text{ a partition of } w\}$, and the set of simple $k\mathcal{S}(w, w)$ -modules $\{L(\lambda) | \lambda \text{ a partition of } w\}$. In this way, $\chi(\lambda)$ corresponds under (non-twisted) Schur-Weyl duality to the character χ^λ of Σ_w . In addition, $\chi(\lambda)$ has a single composition factor isomorphic to $L(\lambda)$, and for any other composition factor $L(\mu)$, we have $\lambda \triangleright \mu$.

Let $\mathcal{A} = End_{\mathcal{E}}(M)$. Then \mathcal{A} is \mathcal{O} -free, and $K\mathcal{A} \cong K\mathbf{B}_{\rho,w}^\Sigma/KI$ is Morita equivalent to the Schur algebra $K\mathcal{S}(w, w)$, by theorem 90. By convention, we match by this Morita equivalence the character $\chi(\lambda)$ of $K\mathcal{S}(w, w)$ with the character $\chi^{[\emptyset, \dots, \emptyset, \lambda']}$ of $K\Sigma_v$.

Theorem 90 also informs us that $k\mathcal{A} = k\mathbf{B}_{\rho,w}^\Sigma/kI$ is Morita equivalent to the Schur algebra $k\mathcal{S}(w, w)$. How do simple modules correspond ?

Proposition 103 *Under the Morita equivalence between $k\mathcal{S}(w, w)$ and $k\mathbf{B}_{\rho,w}^\Sigma/kI$ of theorem 90, the simple module $L(\lambda)$ corresponds to $D^{[\emptyset, \dots, \emptyset, \lambda']}$.*

Proof:

For λ, μ partitions of w , let the multiplicities m_μ^λ be defined by,

$$Ind_{\Sigma_\lambda}^{\Sigma_w} K = \sum_{\mu} m_\mu^\lambda \cdot \chi^\mu.$$

We know for that m_μ^λ is zero for all partitions $\lambda \triangleright \mu$ of w , and that $m_\lambda^\lambda = 1$ for all partitions λ of w .

The right $K\Sigma_w \times \Sigma_r$ -module $f_\lambda KM$ has character,

$$\dim(\chi^\rho) \sum_{\mu} m_\mu^\lambda (\chi^{\mu'} \otimes \chi^\rho).$$

Consider $\text{Hom}_{\mathcal{E}}(M, f_\lambda M)$, a left \mathcal{A} -module, which has character

$$\dim(\chi^\rho) \sum_{\mu} m_\mu^\lambda \chi^{[\emptyset, \dots, \emptyset, \mu']},$$

Reduced modulo p , this is the $k\mathcal{A}$ -module,

$$\text{Hom}_{\Sigma_w \times \Sigma_r}(kM, f_\lambda kM),$$

which via Morita equivalence, corresponds to $\dim(\chi^\rho)$ copies of the $k\mathcal{S}(w, w)$ -module,

$$E^{\otimes r \#} . (\sum_{\sigma \in \Sigma_\lambda} \text{sgn}(\sigma) \sigma).$$

In other words, we have $\dim(\chi^\rho)$ copies of the $k\mathcal{S}(w, w)$ -module,

$$E^{\otimes r} . (\sum_{\sigma \in \Sigma_\lambda} \sigma).$$

This module is isomorphic to a p -modular reduction of the $K\mathcal{S}(w, w)$ -module with character $\dim(\chi^\rho) \sum_{\mu} m_\mu^\lambda \cdot \chi(\mu)$.

Induction according to the dominance ordering now implies that the p -modular reduction of an $K\mathcal{S}(w, w)$ -module with character $\chi(\lambda)$, mapped under Morita equivalence to a $k\mathcal{A}$ -module, has the same composition factors as a p -modular reduction of a $K\Sigma_v$ -module with character $\chi^{[\emptyset, \dots, \emptyset, \lambda']}$. The lower unitriangularity of the decomposition matrices of $k\mathcal{S}(w, w)$ and $k\Sigma_v$ implies the result. \square

Corollary 104 *The decomposition matrix of $k\mathcal{S}(w, w)$ is a submatrix of the decomposition matrix of $k\mathbf{B}_{\rho, w}^\Sigma$, where row and column λ corresponds to row and column $[\emptyset, \dots, \emptyset, \lambda']$. \square*

Remark 105 It is not possible that $k\Sigma_v / \text{Ann}(k\Sigma_v e) \cong \text{End}_{kNf}(k\Sigma_v e)$ when $p = 2$.

Because on one hand (by [51], 1.14.5) the k -dimension of $\text{End}_{ek\Sigma_v e}(k\Sigma_v e)$ is at least as great as the K -dimension of $\text{End}_{KNf}(K\Sigma_v e) \cong b_\rho(K\Sigma_v)$ (this

isomorphism still holds when $w \geq p$, by the character calculation in [11]). On the other hand, the dimension of $k\Sigma_v/\text{Ann}(k\Sigma_v e)$ is strictly smaller than the dimension of $K\mathbf{B}_{\rho,w}^\Sigma$, because there are elements of $k\mathbf{B}_{\rho,w}^\Sigma$ which act on $k\Sigma_v e$ as zero. For example, let $xk\Sigma_v$ be a simple right $k\mathbf{B}_{\rho,w}^\Sigma$ -module, not in the top of $ek\Sigma_v$ (such exist, since $ek\Sigma_v e$ and $k\mathbf{B}_{\rho,w}^\Sigma$ are not Morita equivalent). In this situation, $xk\Sigma_v e \cong \text{Hom}(ek\Sigma_v, xk\Sigma_v) = 0$.

Remark 106

(a) The modules in the top (and socle) of ${}_{\Sigma_v}kM$ are those $D^{[\emptyset, \dots, \emptyset, \lambda]}$'s such that λ is p -regular, by theorem 90 and proposition 103.

(b) The Young module summands of ${}_{\Sigma_v}kM$ are those $Y^{[\emptyset, \dots, \emptyset, \lambda]}$'s such that λ is p -regular, by theorem 90 and proposition 103.

Question 107 *The following are open:*

When $p = 2$, what are the summands of $k\Sigma_v e_N$? What are their vertices and what are their sources ?

When $p = 2$, what is the vertex of ${}_{\Sigma_v}k\Sigma_v e_N$? What is the source ?

Chapter VI

Ringel duality inside Rock blocks of symmetric groups.

We prove that a Rock block of a symmetric group, of arbitrary defect, possesses a family of internal symmetries, given as Ringel dualities between various subquotients (theorem 123). Since this sequence of symmetries resembles J.A. Green's walk around the Brauer tree [37], we name it "a walk along the abacus".

A criterion for Ringel duality.

Let (K, \mathcal{O}, k) be an l -modular system. We prove a sufficient condition for Ringel duality between two split quasi-hereditary \mathcal{O} -algebras.

Let R be a commutative Noetherian ring. Cline, Parshall, and Scott have defined split quasi-hereditary R -algebras ([17], 3.2). For example, the Schur algebra $\mathcal{S}(n, r)$, defined over R , is a split quasi-hereditary algebra, with respect to the poset $\Lambda(n, r)$. (see [17], 3.7).

More generally, for Γ an ideal of $\Lambda(n, r)$, and Ω a coideal of $\Lambda(n, r)$, the generalised Schur algebra $\mathcal{S}(\Gamma \cap \Omega)$ is a split quasi-hereditary subquotient of $\mathcal{S}(n, r)$, with respect to the poset $\Gamma \cap \Omega$.

Definition 108 *For a split quasi-hereditary \mathcal{O} -algebra A , let us define a K - k -tilting module T to be a finitely generated A -order which is a tilting module over K , as well as a tilting module over k .*

The following result resembles M. Broué's theorem 71, which gives sufficient conditions for a Morita equivalence between symmetric \mathcal{O} -algebras:

Theorem 109 *Let A, B be split quasi-hereditary algebras over \mathcal{O} with respect to posets Λ, Υ . Suppose that KA, KB are semisimple. Let T be an A - B -bimodule which is a K - k -tilting module at the same time as a left A -module and as a right B -module. Suppose that the functors*

$$KT \otimes_{KB} - : KB\text{-mod} \rightarrow KA\text{-mod}$$

$$- \otimes_{KA} KT : \text{mod} - KA \rightarrow \text{mod} - KB$$

are equivalences of categories, such that the resulting bijections of irreducible modules define order-reversing maps between Λ and Υ . Then kT defines a Ringel duality between kA and kB^{op} . Furthermore, $B \cong \text{End}_A(T)$, and $A \cong \text{End}_B(T)$.

Proof:

Let $\{P(\mu), \mu \in \Upsilon\}$ be the set of non-isomorphic principal indecomposable B -modules. Let Ψ be the order-reversing map from Υ to Λ defined by KT .

The A -module T has $|\Upsilon|$ distinct summands $T \otimes_B P(\mu)$. Viewed over K , such a summand has a single composition factor $K\Delta(\Psi\mu)$, and all other composition factors $K\Delta(\lambda), \lambda < \Psi\mu$. This is because Ψ is order-reversing. Viewed over k , such a summand has a single composition factor $L(\Psi\mu)$, and all other composition factors $L(\lambda), \lambda < \Psi\mu$. In other words, $kT \otimes_{kB} P(\mu)$ is the indecomposable tilting module $T(\Psi\mu)$ for kA . Thus, kT is a full tilting module as a left kA -module.

Likewise, kT is a full tilting module as a right kB -module. It follows that kA and kB act faithfully on kT .

Note that there are natural isomorphisms $KB \cong \text{End}_{KA}(KT)$ and $KA \cong \text{End}_{KB}(KT)$ of algebras.

The endomorphism ring $\text{End}_A(T)$ is an \mathcal{O} -order, so that $k\text{End}_A(T)$ injects into $\text{End}_{kA}(kT)$, and such that $K\text{End}_A(T)$ surjects onto $\text{End}_{KA}(KT)$, which is isomorphic to KB . The dimension of $\text{End}_{kA}(kT)$ is given ([27], A.2.2(ii)) by the formula,

$$\sum_{\nu} [kT : \Delta_k(\nu)][kT : \nabla_k(\nu)].$$

The dimension of $\text{End}_{KA}(KT)$ is given by the formula,

$$\sum_{\nu} [KT : \Delta_K(\nu)][KT : \nabla_K(\nu)].$$

It follows from the equality of these formulae, that the rank of $\text{End}_A(T)$ is equal to the rank of B , and $k\text{End}_A(T) \cong \text{End}_{kA}(kT)$. Indeed, $k\text{End}_A(T) \cong \text{End}_{kA}(kT) \cong kB$, since kB acts faithfully on kT .

We have now verified that kA and kB are in Ringel duality. Since the natural map from kB to $k\text{End}_A(T)$ is an isomorphism, B is \mathcal{O} -pure in $\text{End}_A(T)$, and so $B \cong \text{End}_A(T)$. Likewise, $A \cong \text{End}_B(T)$. This completes the proof of theorem 109. \square

Combinatorial preliminaries.

Let p be a prime number and w any natural number. Let $\rho = \rho(p, w)$ be a minimal Rouquier core. Thus, in an abacus presentation, ρ has precisely $w - 1$ more beads on the i^{th} runner than on the $i - 1^{\text{th}}$ runner.

Lemma 110 *Let $\tilde{\rho}$ be an arbitrary Rouquier core. Then $\mathcal{OB}_{\tilde{\rho}, w}^{\Sigma}$ is Morita equivalent to $\mathcal{OB}_{\rho, w}^{\Sigma}$. The resulting correspondence of partitions preserves p -quotients.*

Proof:

Scopes' isometries [67] correspond to the motion of k beads one runner leftwards on the abacus, where $k \geq w$. Suppose that $\tilde{\rho}$ has $N_i \geq (w-1)$ more beads on runner i than on runner $i-1$. Suppose that $N_j > (w-1)$. Then there is a Scopes isometry which moves those N_j beads from runner j to runner $j-1$, which may be followed by a Scopes isometry which moves $N_j + N_{j+1}$ beads from runner $j+1$ to runner j, \dots , which may be followed by a Scopes isometry which moves $N_j + N_{j+1} + \dots + N_{p-1}$ beads from runner $p-1$ to runner $p-2$. If $j > 1$, then the result is a p -core with $N'_{j-1} > (w-1)$ more beads on runner $j-1$ than on runner $j-2$.

We may thus proceed with a further $p-j$ Scopes isometries, moving beads from runners $j-1, \dots, p-2$ leftwards. Following this procedure to its natural conclusion (always pushing beads leftwards), and at last circulating the ordering of runners on the abacus so that runner 0 becomes runner j , we obtain a p -core $\tilde{\rho}_0$, smaller than $\tilde{\rho}$ which still satisfies the condition that there are $\geq (w-1)$ more beads on runner i than on runner $i-1$. By induction, lemma 110 is proven. \square

Let

$$\mathcal{I} = \{\lambda \mid \lambda \text{ has core } \rho \text{ and weight } w\}$$

be the indexing poset of $k\mathbf{B}_{\rho,w}^S$. We wish to describe the dominance order on \mathcal{I} . To $\lambda \in \mathcal{I}$ with p -quotient $[\lambda^0, \lambda^1, \dots, \lambda^{p-1}]$, let us associate an element $\langle \lambda \rangle \in \mathbb{N}^{wp}$, given by

$$(\lambda_1^{p-1}, \lambda_2^{p-1}, \dots, \lambda_w^{p-1}, \lambda_1^{p-2}, \lambda_2^{p-2}, \dots, \lambda_w^{p-2}, \dots, \lambda_1^0, \lambda_2^0, \dots, \lambda_w^0).$$

Let us place the dominance order on \mathbb{N}^{wp} . We then have:

Proposition 111 *Let $\lambda, \mu \in \mathcal{I}$. In the dominance order, $\mu \trianglelefteq \lambda$ if and only if $\langle \mu \rangle \trianglelefteq \langle \lambda \rangle$.*

Proof:

Following lemma 110, and applying a series of Scopes isometries (which preserve the dominance order), we may replace ρ by $\tilde{\rho}$, where $\tilde{\rho}$ has at least $N > 2w$ beads on runner i than runner $i-1$ (for $i = 1, \dots, p-1$). Let

$$\tilde{\mathcal{I}} = \{\lambda \mid \lambda \text{ has core } \tilde{\rho}, \text{ and weight } w\}.$$

Suppose that $\lambda \triangleleft \mu$ are neighbours in $\tilde{\mathcal{I}}$. There is a sequence $\lambda = \lambda_0 \triangleleft \lambda_1 \triangleleft \dots \triangleleft \lambda_m = \mu$, where the Young diagram of λ_{j-1} is obtained from the Young diagram of λ_j by removing a box and placing it lower in the diagram (for

$j = 1, \dots, m$). We may assume that no box is removed from the core in this sequence of motions.

Thus, at each step we must remove a box from one of the p -hooks which has been added to $\tilde{\rho}$ to create μ . Since $\tilde{\rho}$ has N more beads on runner i than runner $i - 1$ (for $i = 1, \dots, p - 1$), we must actually remove an entire p -hook if we are to end our sequence in $\tilde{\mathcal{I}}$. Correspondingly, we must add an entire p -hook when we add boxes. The new additional p -hook must appear lower in the Young diagram than the old removed p -hook.

On an abacus, the corresponding motion looks as follows: move a single bead one place higher on its runner, move a second bead (necessarily above the first bead) one place lower on *its* runner.

This corresponds precisely to removing one box from the Young diagram of $\langle \lambda \rangle$, and replacing it lower in the Young diagram to obtain $\langle \mu \rangle$. Thus, $\langle \lambda \rangle \triangleleft \langle \mu \rangle$.

Working backwards, we find conversely that $\langle \lambda \rangle \triangleleft \langle \mu \rangle$ implies $\mu \triangleleft \lambda$. \square

We may deduce from the above proposition a number of combinatorial results concerning ideals and coideals of the poset \mathcal{I} , ordered by the dominance ordering.

Definition 112 For natural numbers a_1, \dots, a_{p-1} such that $\sum_{i=0}^{p-1} a_i = w$, let

$$\mathcal{I}_{(a_0, a_1, \dots, a_{p-1})} = \left\{ \lambda \in \mathcal{I} \mid \begin{array}{l} \lambda \trianglelefteq \mu \text{ for some } \mu \in \mathcal{I} \text{ with } p\text{-quotient} \\ [\mu^0, \dots, \mu^{p-1}], \text{ such that } |\mu^i| = a_i \end{array} \right\}$$

$$\mathcal{J}_{(a_0, a_1, \dots, a_{p-1})} = \left\{ \lambda \in \mathcal{I} \mid \begin{array}{l} \mu \trianglelefteq \lambda \text{ for some } \mu \in \mathcal{I} \text{ with } p\text{-quotient} \\ [\mu^0, \dots, \mu^{p-1}], \text{ such that } |\mu^i| = a_i \end{array} \right\}$$

For $i = 0, \dots, p - 1$, let

$$\mathcal{I}_i = \mathcal{I}_{(0, \dots, 0, w, 0, \dots, 0)}$$

$$\mathcal{J}_i = \mathcal{I}_{(0, \dots, 0, w, 0, \dots, 0)},$$

where the w appears as the i^{th} entry in $(0, \dots, 0, w, 0, \dots, 0)$. Let

$$\mathcal{I}_{\text{res}} = \mathcal{I}_{p-2}, \quad \mathcal{I}_{\text{unres}} = \mathcal{I} - \mathcal{I}_{\text{res}} \quad \mathcal{I}_{\text{reg}} = \mathcal{J}_1, \quad \mathcal{I}_{\text{sing}} = \mathcal{I} - \mathcal{I}_{\text{reg}}.$$

Following these definitions, proposition 111 has a number of corollaries, which are easily checked:

Corollary 113 *These subsets of \mathcal{I} are ideals:*

$$\mathcal{I}_{(a_0, a_1, \dots, a_{p-1})}, \mathcal{I}_i, \mathcal{I}_{res}, \mathcal{I}_{sing}.$$

These subsets of \mathcal{I} are coideals:

$$\mathcal{J}_{(a_0, a_1, \dots, a_{p-1})}, \mathcal{J}_i, \mathcal{I}_{unres}, \mathcal{I}_{reg}. \square$$

Corollary 114 *For $i = 0, \dots, p-1$,*

$$\begin{aligned} \mathcal{I}_i &= \left\{ \lambda \in \mathcal{I} \mid \begin{array}{l} \text{with } p\text{-quotient } [\lambda^0, \dots, \lambda^{p-1}], \text{ such that} \\ |\lambda^{i+1}| = |\lambda^{i+2}| = \dots = |\lambda^{p-1}| = 0 \end{array} \right\} \\ \mathcal{J}_i &= \left\{ \lambda \in \mathcal{I} \mid \begin{array}{l} \text{with } p\text{-quotient } [\lambda^0, \dots, \lambda^{p-1}], \text{ such that} \\ |\lambda^0| = |\lambda^1| = \dots = |\lambda^{i-1}| = 0 \end{array} \right\}. \square \end{aligned}$$

Corollary 115

$$\mathcal{I}_{res} = \{ \lambda \in \mathcal{I} \mid \lambda \text{ is } p\text{-restricted} \}.$$

$$\mathcal{I}_{unres} = \{ \lambda \in \mathcal{I} \mid \lambda \text{ is } p\text{-nonrestricted} \}.$$

$$\mathcal{I}_{reg} = \{ \lambda \in \mathcal{I} \mid \lambda \text{ is } p\text{-regular} \}.$$

$$\mathcal{I}_{sing} = \{ \lambda \in \mathcal{I} \mid \lambda \text{ is } p\text{-singular} \}. \square$$

Corollary 116 *The intersection $\mathcal{I}_{(a_0, a_1, \dots, a_{p-1})} \cap \mathcal{J}_{(a_0, a_1, \dots, a_{p-1})}$ is equal to the set,*

$$\{ \lambda \in \mathcal{I} \mid \lambda \text{ has } p\text{-quotient } [\lambda^0, \dots, \lambda^{p-1}], \text{ where } |\lambda^i| = a_i \}. \square$$

The intersection of ideals in the above lemma is analogous to the classical intersection $\{ \mu \trianglelefteq \lambda \} \cap \{ \mu \trianglerighteq \lambda \} = \{ \lambda \}$. Whilst this classical intersection may be used to index the characters of symmetric groups by partitions, we use the intersection of ideals above to study Rock blocks runner by runner.

Definition 117 *For natural numbers a_0, \dots, a_{p-1} such that $\sum_{i=0}^{p-1} a_i = w$, let*

$$\mathcal{K}_{(a_0, \dots, a_{p-1})} = \mathcal{I}_{(a_0, \dots, a_{p-1})} \cap \mathcal{J}_{(a_0, \dots, a_{p-1})} =$$

$$\{ \lambda \in \mathcal{I} \mid \lambda \text{ has } p\text{-quotient } [\lambda^0, \dots, \lambda^{p-1}], \text{ where } |\lambda^i| = a_i \}.$$

For $i = 0, \dots, p-1$, let

$$\mathcal{K}_i = \mathcal{I}_i \cap \mathcal{J}_i =$$

$$\left\{ \lambda \in \mathcal{I} \mid \begin{array}{l} \lambda \text{ has } p\text{-quotient } [\lambda^0, \dots, \lambda^{p-1}], \text{ such that} \\ |\lambda^0| = \dots = |\lambda^{i-1}| = |\lambda^{i+1}| = \dots = |\lambda^{p-1}| = 0, |\lambda^i| = w \end{array} \right\}.$$

Quasi-hereditary subquotients of $\mathcal{OB}_{\rho,w}^\Sigma$.

We introduce pairs of quasi-hereditary quotients of $\mathcal{OB}_{\rho,w}^\Sigma$, which are isomorphic under the signature automorphism.

Let $\omega = (1^v)$, a partition of v . Let $\mathcal{S}(v, v)$ be the Schur algebra associated to polynomial representations of GL_v of degree v , defined over \mathcal{O} . Recall that $\xi_\omega \mathcal{S}(v, v) \xi_\omega$ is isomorphic to \mathcal{OS}_v (theorem 23).

To an ideal Γ of \mathcal{I} , let us associate the ideal X_Γ of $\mathcal{S}(v, v)$, the quotient by which, is the generalised Schur algebra, $\mathcal{S}(\Gamma)$. Let $I_\Gamma = \xi_\omega b_{\rho,w}^\mathcal{S} X_\Gamma \xi_\omega$ be the corresponding ideal of $\mathcal{OB}_{\rho,w}^\Sigma$.

Lemma 118 *Suppose that Γ is an ideal of \mathcal{I}_{res} . Then $\mathcal{OB}_{\rho,w}^\Sigma/I_\Gamma$ is a quasi-hereditary algebra, with indexing poset Γ . The decomposition matrix of the algebra $\mathcal{OB}_{\rho,w}^\Sigma/I_\Gamma$ is the square submatrix of the decomposition matrix of $\mathcal{OB}_{\rho,w}^\Sigma$, whose rows are indexed by elements of Γ .*

Proof:

The generalised Schur algebra $\mathcal{OB}_{\rho,w}^\mathcal{S}/X_\Gamma$ is a quasi-hereditary algebra, whose indexing poset is Γ .

Over the field k , the idempotent ξ_ω sends to zero precisely those simple modules indexed by unrestricted partitions. Thus, $(\mathcal{OB}_{\rho,w}^\mathcal{S}/X_\Gamma)\xi_\omega$ is a progenerator for $\mathcal{OB}_{\rho,w}^\Sigma/I_\Gamma$. It follows that,

$$\xi_\omega(\mathcal{OB}_{\rho,w}^\mathcal{S}/X_\Gamma)\xi_\omega \cong \mathcal{OB}_{\rho,w}^\Sigma/\xi_\omega X_\Gamma \xi_\omega = \mathcal{OB}_{\rho,w}^\Sigma/I_\Gamma,$$

is Morita equivalent to $\mathcal{OB}_{\rho,w}^\Sigma/I_\Gamma$. \square

Let a_1, \dots, a_{p-1} be natural numbers such that $\sum_{i=1}^{p-1} a_i = w$.

We write $I_{(a_1, \dots, a_{p-1})}$ for I_Γ , where Γ is the ideal $\mathcal{I}_{(a_1, \dots, a_{p-1}, 0)}$ in \mathcal{I} .

Thus, $\mathcal{OB}_{\rho,w}^\Sigma/I_{(a_1, \dots, a_{p-1})}$ is a quasi-hereditary algebra whose poset is the ideal $\mathcal{I}_{(a_1, \dots, a_{p-1}, 0)}$, and $\{D_\lambda | \lambda \in \mathcal{I}_{(a_1, \dots, a_{p-1}, 0)}\}$ is a complete set of non-isomorphic simple $k\mathcal{B}_{\rho,w}^\Sigma/I_{(a_1, \dots, a_{p-1})}$ -modules.

So long as $i = 1, \dots, p-1$, let us write I_i for $I_{(0, \dots, 0, w, 0, \dots, 0)}$, where w appears as the $i-1^{th}$ entry in $(0, \dots, 0, w, 0, \dots, 0)$.

Thus, $\mathcal{OB}_{\rho,w}^\Sigma/I_i$ is a quasi-hereditary algebra whose poset is \mathcal{I}_{i-1} .

Let us write I_{unres} for I_{p-2} . Thus, $\mathcal{OB}_{\rho,w}^\Sigma/I_{unres}$ is a quasi-hereditary algebra whose poset is \mathcal{I}_{res} .

Let $\Omega \subset \mathcal{I}$ be a coideal. Let x_Ω be an idempotent in $\mathcal{O}\Sigma_v$, such that $\mathcal{O}\Sigma_v x_\Omega$ is a maximal summand of $\mathcal{O}\Sigma_v$, whose indecomposable summands have tops in $\{D_\lambda | \lambda \in \mathcal{I}_{res} \cap \Omega\}$.

Lemma 119 *Suppose that Ω is a coideal of \mathcal{I} , and that Γ is an ideal of \mathcal{I}_{res} . Let $A_{\Gamma \cap \Omega} = x_\Omega(\mathcal{O}\mathbf{B}_{\rho,w}^\Sigma / I_\Gamma)x_\Omega$. Then $A_{\Gamma \cap \Omega}$ is a quasi-hereditary algebra, with indexing poset $\Gamma \cap \Omega$.*

Proof:

$A_{\Gamma \cap \Omega}$ is Morita equivalent to the generalised Schur algebra with indexing poset $\Gamma \cap \Omega$, via the bimodule $\xi_\Omega(\mathcal{S}(v, v) / J_{\Gamma \cap \Omega})\xi_\omega.x_\Omega$. \square

Suppose that a_1, \dots, a_{p-1} are natural numbers such that $\sum_{i=1}^{p-1} a_i = w$.

We write $A_{(a_1, \dots, a_{p-1})}$ for $A_{\Gamma \cap \Omega}$, where $\Omega = \mathcal{J}_{(a_1, \dots, a_{p-1}, 0)}$, and $\Gamma = \mathcal{I}_{(a_1, \dots, a_{p-1}, 0)}$. Indeed, $A_{(a_1, \dots, a_{p-1})}$ is a quasi-hereditary algebra whose poset is $\mathcal{K}_{(a_1, \dots, a_{p-1}, 0)}$, by corollary 116.

So long as $i = 1, \dots, p-1$, we write A_i for $A_{(0, \dots, 0, w, 0, \dots, 0)}$, where w appears as the $i-1^{th}$ entry in $(0, \dots, 0, w, 0, \dots, 0)$. Thus, A_i is a quasi-hereditary algebra whose poset is \mathcal{K}_{i-1} .

For natural numbers a_1, \dots, a_{p-1} such that $\sum_{i=1}^{p-1} a_i = w$, let $J_{(a_1, \dots, a_{p-1})}$ be the ideal $I_{(a_{p-1}, \dots, a_1)}^\#$. Thus, $\mathcal{O}\mathbf{B}_{\rho,w}^\Sigma / J_{(a_1, \dots, a_{p-1})}$ is a quasi-hereditary algebra whose poset is $\mathcal{J}_{(0, a_1, \dots, a_{p-1})}^{op}$.

We write $\{D^\lambda | \lambda \in \mathcal{J}_{(0, a_1, \dots, a_{p-1})}\}$ for the set of simple $k\mathbf{B}_{\rho,w}^\Sigma / J_{(a_1, \dots, a_{p-1})}$ -modules.

For $i = 1, \dots, p-1$, let $J_i = I_{p-i}^\#$. Thus, $\mathcal{O}\mathbf{B}_{\rho,w}^\Sigma / J_i$ is a quasi-hereditary algebra whose poset is \mathcal{J}_i^{op} .

Let $J_{sing} = I_{unres}^\#$. Thus, $\mathcal{O}\mathbf{B}_{\rho,w}^\Sigma / J_{sing}$ is a quasi-hereditary algebra whose poset is \mathcal{I}_{reg}^{op} .

Let $y_{(a_1, \dots, a_{p-1})} = x_{(a_{p-1}, \dots, a_1)}^\#$. Let

$$B_{(a_1, \dots, a_{p-1})} = y_{(a_1, \dots, a_{p-1})} (\mathcal{O}\mathbf{B}_{\rho,w}^\Sigma / J_{(a_1, \dots, a_{p-1})}) y_{(a_1, \dots, a_{p-1})}.$$

Thus, $B_{(a_1, \dots, a_{p-1})}$ is a quasi-hereditary algebra whose poset is $\mathcal{K}_{(0, a_1, \dots, a_{p-1})}^{op}$.

So long as $i = 1, \dots, p-1$, let us write B_i for $B_{(0, \dots, 0, w, 0, \dots, 0)}$, where w appears as the i^{th} entry in $(0, \dots, 0, w, 0, \dots, 0)$. Thus, B_i is a quasi-hereditary algebra whose poset is \mathcal{K}_i^{op} .

Remark 120 Simple $kA_{(a_1, \dots, a_{p-1})}$ -modules are in natural one-one correspondence with the set $\{D_\lambda \mid \lambda \in \mathcal{K}_{(a_1, \dots, a_{p-1}, 0)}\}$. By theorem 61, this set is equal to the set $\{D^\lambda \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-1})}\}$.

Simple $kB_{(a_1, \dots, a_{p-1})}$ -modules are also in natural one-one correspondence with the set $\{D^\lambda \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-1})}\}$.

By balancing the algebras $\mathcal{OB}_{\rho, w}^\Sigma / I_{unres}$ and $\mathcal{OB}_{\rho, w}^\Sigma / J_{sing}$ on the Mullineux map, we reveal Ringel dualities between different runners of $\mathcal{OB}_{\rho, w}^\Sigma$, in the following section.

Walking along the abacus.

Let $p \geq 3$. For $\sum_{i=1}^{p-2} a_i = w$, consider the \mathcal{O} -lattice

$$N_{(a_1, \dots, a_{p-2})} = x_{(0, a_1, \dots, a_{p-2})} \mathcal{O} \Sigma v y_{(a_1, \dots, a_{p-2}, 0)}.$$

In this section we prove that $N_{(a_1, \dots, a_{p-2})}$ provides a Ringel duality between the quasi-hereditary subquotients $A_{(0, a_1, \dots, a_{p-2})}$ and $B_{(a_1, \dots, a_{p-2}, 0)}$ of $\mathcal{OB}_{\rho, w}^\Sigma$.

These Ringel dualities should be viewed as internal symmetries of the Rock block. For simple $kA_{(0, a_1, \dots, a_{p-2})}$ -modules are in natural correspondence with simple $k\mathbf{B}_{\rho, w}^\Sigma$ -modules D^λ indexed by elements of $\mathcal{K}_{(0, 0, a_1, \dots, a_{p-2})}$. At the same time, simple $kB_{(a_1, \dots, a_{p-2}, 0)}$ -modules are in natural correspondence with simple $k\mathbf{B}_{\rho, w}^\Sigma$ -modules D^μ indexed by elements of $\mathcal{K}_{(0, a_1, \dots, a_{p-2}, 0)}$.

These symmetries therefore enable us to translate module-theoretic information along the abacus.

Here's a technical lemma:

Lemma 121 Suppose that $p \geq 3$. Let $\sum_{i=1}^{p-2} a_i = w$. Then

$$J_{sing} \cdot x_{(0, a_1, \dots, a_{p-2})} = 0,$$

$$J_{(a_1, \dots, a_{p-2}, 0)} \cdot x_{(0, a_1, \dots, a_{p-2})} = x_{(0, a_1, \dots, a_{p-2})} \cdot J_{(a_1, \dots, a_{p-2}, 0)} = 0,$$

$$I_{unres} \cdot y_{(a_1, \dots, a_{p-2}, 0)} = 0,$$

$$I_{(0, a_1, \dots, a_{p-2})} \cdot y_{(a_1, \dots, a_{p-2}, 0)} = y_{(a_1, \dots, a_{p-2}, 0)} \cdot I_{(0, a_1, \dots, a_{p-2})} = 0.$$

Proof:

The character of $\mathcal{O} \Sigma v \cdot x_{(0, a_1, \dots, a_{p-2})}$ has irreducible components χ^λ , where λ lies in the coideal $\mathcal{J}_{(0, a_1, \dots, a_{p-2}, 0)}$ of \mathcal{I} . This character has no components which are p -singular, and no components which lie in $\mathcal{I} - \mathcal{J}_{(0, a_1, \dots, a_{p-2}, 0)}$.

Over K , the ideal J_{sing} is equal to the Wedderburn component of $K\mathbf{B}_{\rho, w}^\Sigma$ with p -singular components, whilst $J_{(a_1, \dots, a_{p-2}, 0)}$ is equal to the Wedderburn

component with components in $\mathcal{I} - \mathcal{J}_{(0,a_1,\dots,a_{p-2},0)}$. This, along with the fact that irreducible characters for symmetric groups are self-dual, proves the first two parts of the lemma. The third and fourth parts follow analogously. \square

We now show that $N_{(a_1,\dots,a_{p-2})}$ is an $A_{(0,a_1,\dots,a_{p-2})}$ - $B_{(a_1,\dots,a_{p-2},0)}$ -bimodule:

Lemma 122 *The kernel of the natural map from $x_{(0,a_1,\dots,a_{p-2})} \mathcal{O}_{\Sigma_v} x_{(0,a_1,\dots,a_{p-2})}$ to $\text{End}(N_{(a_1,\dots,a_{p-2})})$ contains the ideal $x_{(0,a_1,\dots,a_{p-2})} I_{(0,a_1,\dots,a_{p-2})} x_{(0,a_1,\dots,a_{p-2})}$.*

The kernel of the natural map from $y_{(a_1,\dots,a_{p-2},0)} \mathcal{O}_{\Sigma_v} y_{(a_1,\dots,a_{p-2},0)}$ to the ring $\text{End}(N_{(a_1,\dots,a_{p-2})})$, contains the ideal $y_{(a_1,\dots,a_{p-2},0)} J_{(a_1,\dots,a_{p-2},0)} y_{(a_1,\dots,a_{p-2},0)}$.

Thus, $N_{(a_1,\dots,a_{p-2})}$ is an $A_{(0,a_1,\dots,a_{p-2})}$ - $B_{(a_1,\dots,a_{p-2},0)}$ -bimodule.

Proof:

$$\begin{aligned} & x_{(0,a_1,\dots,a_{p-2})} I_{(0,a_1,\dots,a_{p-2})} x_{(0,a_1,\dots,a_{p-2})} \mathcal{O}_{\Sigma_v} y_{(a_1,\dots,a_{p-2},0)} \\ & \subseteq x_{(0,a_1,\dots,a_{p-2})} I_{(0,a_1,\dots,a_{p-2})} y_{(a_1,\dots,a_{p-2},0)} = 0, \end{aligned}$$

by lemma 121. Likewise,

$$\begin{aligned} & x_{(0,a_1,\dots,a_{p-2})} \mathcal{O}_{\Sigma_v} y_{(a_1,\dots,a_{p-2},0)} J_{(a_1,\dots,a_{p-2},0)} y_{(a_1,\dots,a_{p-2},0)} \\ & \subseteq x_{(0,a_1,\dots,a_{p-2})} J_{(a_1,\dots,a_{p-2},0)} y_{(a_1,\dots,a_{p-2},0)} = 0. \square \end{aligned}$$

Theorem 123 (“Walking along the abacus”) *The bimodule $N_{(a_1,\dots,a_{p-2})}$ defines a Ringel duality between $kA_{(0,a_1,\dots,a_{p-2})}$ and $k\mathbf{B}_{(a_1,\dots,a_{p-2},0)}^{op}$.*

Proof:

We first show that $N_{(a_1,\dots,a_{p-2})}$ is a K - k -tilting module both as a left $A_{(0,a_1,\dots,a_{p-2})}$ -module, and as a right $B_{(a_1,\dots,a_{p-2},0)}$ -module.

Let $R \in \{K, k\}$. Recall that under Schur-Weyl duality, Specht modules correspond to costandard modules. Therefore, the costandard modules for $R\mathbf{B}_{\rho,w}^{\Sigma}/I_{unres}$ are those Specht modules indexed by restricted partitions. The costandard modules for $R\mathbf{B}_{\rho,w}^{\Sigma}/I_{(0,a_1,\dots,a_{p-2})}$ are those Specht modules indexed by elements of $\mathcal{I}_{(0,a_1,\dots,a_{p-2},0)}$. The costandard modules for $A_{(0,a_1,\dots,a_{p-2})}$ are those modules $x_{(0,a_1,\dots,a_{p-2})} S$, where S is a Specht module indexed by an element of $\mathcal{K}_{(0,a_1,\dots,a_{p-2},0)}$.

Since $R\Sigma_v y_{(a_1,\dots,a_{p-2},0)}$ is a projective module, it has a filtration by Specht modules. Suppose that S is a Specht module in this filtration. Then S is a costandard module for $R\mathbf{B}_{\rho,w}^{\Sigma}/I_{(0,a_1,\dots,a_{p-2})}$, by lemma 121. Thus, $x_{(0,a_1,\dots,a_{p-2})} S$ is a costandard module for $A_{(0,a_1,\dots,a_{p-2})}$. So $N_{(a_1,\dots,a_{p-2})}$ has a filtration by costandard modules.

Since $R\Sigma_v y_{(a_1, \dots, a_{p-2})}$ is self-dual, it may also be filtered by dual Specht modules. The same argument as above now shows that $N_{(a_1, \dots, a_{p-2})}$ has a filtration by standard modules.

Thus, $N_{(a_1, \dots, a_{p-2})}$ is a left K - k -tilting module for $A_{(0, a_1, \dots, a_{p-2})}$. In the same way, $N_{(a_1, \dots, a_{p-2})}$ is a right K - k -tilting module for $B_{(a_1, \dots, a_{p-2}, 0)}$.

We would like to apply theorem 109.

Recall that the $K\Sigma_v$ - $K\Sigma_v$ -bimodule $K\Sigma_v$ has character, $\bigoplus_{\lambda} \chi^{\lambda} \otimes \chi^{\lambda}$. In this way, the $x_{(0, a_1, \dots, a_{p-2})} K\Sigma_v x_{(0, a_1, \dots, a_{p-2})}^- y_{(a_1, \dots, a_{p-2}, 0)} K\Sigma_v y_{(a_1, \dots, a_{p-2}, 0)}$ -module $KN_{(a_1, \dots, a_{p-2})}$ has character,

$$\bigoplus_{\lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-2}, 0)}} x_{(0, a_1, \dots, a_{p-2})} \cdot \chi^{\lambda} \otimes \chi^{\lambda} \cdot y_{(a_1, \dots, a_{p-2}, 0)}.$$

Note that,

$$\{x_{(0, a_1, \dots, a_{p-2})} \cdot \chi^{\lambda} \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-2}, 0)}\},$$

is a complete set of irreducible left $KA_{(0, a_1, \dots, a_{p-2})}$ -modules. And that,

$$\{\chi^{\lambda} \cdot y_{(a_1, \dots, a_{p-2}, 0)} \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-2}, 0)}^{op}\},$$

is a complete set of irreducible right $KB_{(a_1, \dots, a_{p-2}, 0)}$ -modules.

Thus, $KN_{(a_1, \dots, a_{p-2})}$ induces a Morita equivalence between $KA_{(0, a_1, \dots, a_{p-2})}$ and $KB_{(a_1, \dots, a_{p-2}, 0)}$, which reverses order on the indexing posets. It is now a consequence of theorem 109, that $kA_{(0, a_1, \dots, a_{p-2})}$, and $kB_{(a_1, \dots, a_{p-2}, 0)}^{op}$ are in Ringel duality. \square

Remark 124 Under the Morita equivalence provided by $KN_{(a_1, \dots, a_{p-2})}$, a simple $KA_{(0, a_1, \dots, a_{p-2})}$ -module S_A^{λ} corresponds to a simple $KB_{(a_1, \dots, a_{p-2}, 0)}$ -module S_B^{λ} .

Note 125 Both $A_{(a_1, \dots, a_{p-1})}$ and $B_{(a_1, \dots, a_{p-1})}$ have simple modules in natural correspondence with $\{D^{\lambda} \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-1})}\}$. Let $\mathcal{L}(a_1, \dots, a_{p-1})$ be the Serre subcategory of $k\Sigma_v$ -mod generated by $\{D^{\lambda} \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-1})}\}$.

Chapter VII

James adjustment algebras for Rock blocks of symmetric groups.

Let $k\mathbf{B}_{\rho,w}^{\Sigma}$ be a Rock block of a symmetric group, whose weight is w . We show that there is a nilpotent ideal \mathcal{N} of $k\mathbf{B}_{\rho,w}^{\Sigma}$, such that $k\mathbf{B}_{\rho,w}^{\Sigma}/\mathcal{N}$ is Morita equivalent to a direct sum,

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right),$$

of tensor products of Schur algebras (theorem 132). The decomposition matrix of this quotient is equal to the James adjustment matrix of $k\mathbf{B}_{\rho,w}^{\Sigma}$ [45].

In chapter five, we proved the existence of a quotient of $k\mathbf{B}_{\rho,w}^{\Sigma}$, equivalent to $\mathcal{S}(w, w)$. In this chapter, we show that the Ringel dualities of chapter VI, may be applied simultaneously with the ideas of chapter V, to set up an induction, proving the existence of a quotient $k\mathbf{B}_{\rho,w}^{\Sigma}/\mathcal{N}$, described above.

Although we choose not explicitly to describe its proof here, there exists a generalisation of the main result of this chapter to arbitrary Hecke algebras of type A . Indeed, there exists a nilpotent ideal \mathcal{N} of $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$, such that $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ is Morita equivalent to a direct sum,

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right),$$

of tensor products of *unquantized* Schur algebras. The decomposition matrix of this quotient is equal to the James adjustment matrix of $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$.

The James adjustment algebra of a Hecke algebra.

Let (K, \mathcal{O}, k) be an l -modular system. Let \wp be the maximal ideal of \mathcal{O} , so that $\mathcal{O}/\wp \cong k$. Let $q \in \mathcal{O}$ be a primitive p^{th} root of unity, whose image in k , is non-zero.

This section is devoted to the Hecke algebra $\mathcal{H}_q(\Sigma_n)$. We define a quotient of this algebra, whose representation theory controls the James adjustment matrix of $\mathcal{H}_q(\Sigma_n)$ (see [35]).

Let us label the simple $K\mathcal{H}_q(\Sigma_n)$ -modules,

$$\{D_q^\lambda \mid \lambda \text{ a } p\text{-regular partition of } n\},$$

in contrast with the simple $k\mathcal{H}_q(\Sigma_n)$ -modules, which we label,

$$\{D^\lambda \mid \lambda \text{ a } p\text{-regular partition of } n\}.$$

The algebra $K\mathcal{H}_q(\Sigma_n)$ is far from semisimple in general, and thus has a non-trivial radical. Let \mathcal{ON}_q be the intersection of this radical with the subalgebra $\mathcal{OH}_q(\Sigma_n)$ of $K\mathcal{H}_q(\Sigma_n)$. Thus, \mathcal{ON}_q is equal to the annihilator in $\mathcal{OH}_q(\Sigma_n)$ of all simple $K\mathcal{H}_q(\Sigma_n)$ -modules. The ideal \mathcal{ON}_q is an \mathcal{O} -pure sublattice of $\mathcal{OH}_q(\Sigma_n)$.

The algebra $\mathcal{G}_q(\Sigma_n) = \mathcal{OH}_q(\Sigma_n)/\mathcal{ON}_q$ is an \mathcal{O} -free algebra, whose square decomposition matrix, $([D_q^\lambda : D^\mu])$ is equal to the James adjustment matrix of $\mathcal{H}_q(\Sigma_n)$. We call it the *James adjustment algebra* of $\mathcal{H}_q(\Sigma_n)$. Indeed, we have (cf. [35], 2.3),

$$Dec_\varphi(\mathcal{OH}_q(\Sigma_n)) = Dec_{<t-q>}(K[t]_{(t-q)}\mathcal{H}_t(\Sigma_n)) \times Dec_\varphi(\mathcal{G}_q(\Sigma_n)).$$

Here, we write $Dec_{\mathcal{J}}(A)$ for the decomposition matrix of an algebra A , defined over a ring R , relative to a maximal ideal \mathcal{J} .

Cutting $\mathcal{G}_q(\Sigma_n)$ at a block $b_{\tau,w}^{\mathcal{H}_q}$ of $\mathcal{H}_q(\Sigma_n)$, we obtain the James adjustment algebra of the block, which we denote $\mathcal{G}b_{\tau,w}^{\mathcal{H}_q}$. If $q = 1$, we label this block $\mathcal{G}b_{\tau,w}^\Sigma$.

Preliminaries on adjustment algebras for Rock blocks.

We now concentrate on Rock blocks of symmetric groups. Thus, we assume that the image of q in k , is equal to 1, and k has characteristic $l = p$. And we adopt the notation of chapters 4-6.

Let $x_{(a_1, \dots, a_{p-1}), q}$ be the q -analogue of $x_{(a_1, \dots, a_{p-1})}$, an idempotent of $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$. Let $A_{(a_1, \dots, a_{p-1}), q}$ be the q -analogue of $A_{(a_1, \dots, a_{p-1})}$, and let $B_{(a_1, \dots, a_{p-1}), q}$ be the q -analogue of $B_{(a_1, \dots, a_{p-1})}$. These are subquotients of $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$.

For natural numbers a_i such that $\sum_{i=1}^{p-1} a_i = w$, let $\mathcal{L}_q(a_1, \dots, a_{p-1})$ be the Serre subcategory of $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q} - mod$ generated by simple modules

$$\{D_q^\lambda \mid \lambda \in \mathcal{K}_{(0, a_1, \dots, a_{p-1})}\}.$$

Let $\mathcal{N}_{a_1, \dots, a_{p-1}, q}$ be the ideal of $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$, the quotient by which, is equal to the Wedderburn component of the quotient $K\mathcal{G}_q(\Sigma_v)$ with components in $\mathcal{L}_q(a_1, \dots, a_{p-1})$.

Proposition 126 *There are isomorphisms of algebras,*

$$K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/K\mathcal{N}_{a_1,\dots,a_{p-1},q} \cong KA_{(a_1,\dots,a_{p-1}),q} \cong KB_{(a_1,\dots,a_{p-1}),q}.$$

Proof:

$$\begin{aligned} & K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/K\mathcal{N}_{a_1,\dots,a_{p-1},q} \\ &= x_{(a_1,\dots,a_{p-1}),q} \left(K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/K\mathcal{N}_{a_1,\dots,a_{p-1},q} \right) x_{(a_1,\dots,a_{p-1}),q}, \end{aligned}$$

which is equal to the Wedderburn component of,

$$x_{(a_1,\dots,a_{p-1}),q} \left(K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\text{Radical} \right) x_{(a_1,\dots,a_{p-1}),q},$$

whose simple components correspond to simple objects of $\mathcal{L}_q(a_1, \dots, a_{p-1})$.

But the simple $KA_{(a_1,\dots,a_{p-1}),q}$ -modules are in one-one correspondence with simple objects of $\mathcal{L}_q(a_1, \dots, a_{p-1})$. Thus, $KA_{(a_1,\dots,a_{p-1}),q}$ surjects onto the quotient

$$K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/K\mathcal{N}_{a_1,\dots,a_{p-1},q}.$$

However, $KA_{(a_1,\dots,a_{p-1}),q}$ is semisimple by proposition 137(1), so this surjection is an isomorphism.

Applying $\#$ to this isomorphism, we discover that in addition,

$$K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/K\mathcal{N}_{a_{p-1},\dots,a_1,q} \cong KB_{(a_{p-1},\dots,a_1),q}. \square$$

Quotients of $k\mathbf{B}_{\rho,w}^\Sigma$.

The algebras $A_{(a_1,\dots,a_{p-1})}$ and $B_{(a_1,\dots,a_{p-1})}$ may be realised as quotients of $k\mathbf{B}_{\rho,w}^\Sigma$, and not merely as subquotients. This will be shown in general in proposition 135, and is crucial to the proof of our main result, theorem 132. As a preliminary, in this section we prove that $A_{(0,a_2,\dots,a_{p-2},0)}$ and $B_{(0,a_2,\dots,a_{p-2},0)}$ may be realised as quotients of $k\mathbf{B}_{\rho,w}^\Sigma$.

We also prove here a baby version of theorem 132, so you may see how these ideas fit together with those of chapter 6 to provide information on $k\mathbf{B}_{\rho,w}^\Sigma - \text{mod}$.

Lemma 127 *Suppose that $p \geq 3$. Let $\sum_{i=1}^{p-2} a_i = w$.*

Then

$$\begin{aligned} & k\mathbf{B}_{\rho,w}^\Sigma/I_{(a_1,\dots,a_{p-2},0)} - \text{mod} \cong \\ & \{M \in k\Sigma_v - \text{mod} \mid M \text{ has composition factors } D^\lambda, \lambda \in \mathcal{I}_{(0,a_1,\dots,a_{p-2},0)} \cap \mathcal{I}_{\text{reg}}\}. \end{aligned}$$

Let $i = 2, \dots, p-1$. Then

$$\begin{aligned} & k\mathbf{B}_{\rho,w}^\Sigma/J_{(0,a_1,\dots,a_{p-2})} - \text{mod} \cong \\ & \{M \in k\Sigma_v - \text{mod} \mid M \text{ has composition factors } D^\lambda, \lambda \in \mathcal{J}_{(0,0,a_1,\dots,a_{p-2})}\}. \end{aligned}$$

Proof:

Let $p \geq 3$, and $\sum_{i=1}^{p-2} a_i = w$. Then,

$$\begin{aligned} & \{M \in k\Sigma_v - \text{mod} \mid M \text{ has simple factors } D^\lambda, \lambda \in \mathcal{I}_{(0,a_1,\dots,a_{p-2},0)} \cap \mathcal{I}_{\text{reg}}\} \\ & \subseteq \{M \in k\Sigma_v - \text{mod} \mid M \text{ is generated by } k\Sigma_v y_{(a_1,\dots,a_{p-2},0)}\} \cong \\ & \{M \in k\mathbf{B}_{\rho,w}^\Sigma / I_{\text{unres}} - \text{mod} \mid M \text{ is generated by } k\Sigma_v y_{(a_1,\dots,a_{p-2},0)}\}, \end{aligned}$$

where the latter isomorphism is by lemma 121. Thus,

$$\begin{aligned} & \{M \in k\Sigma_v - \text{mod} \mid M \text{ has simple factors } D^\lambda, \lambda \in \mathcal{I}_{(0,a_1,\dots,a_{p-2},0)} \cap \mathcal{I}_{\text{reg}}\} \\ & = \left\{ M \in k\mathbf{B}_{\rho,w}^\Sigma / I_{\text{unres}} - \text{mod} \mid \begin{array}{l} M \text{ has simple factors } D^\lambda, \\ \lambda \in \mathcal{I}_{(0,a_1,\dots,a_{p-2},0)} \cap \mathcal{I}_{\text{reg}} \end{array} \right\} \\ & = \left\{ M \in k\mathbf{B}_{\rho,w}^\Sigma / I_{\text{unres}} - \text{mod} \mid \begin{array}{l} M \text{ has simple factors } D_\lambda, \\ \lambda \in \mathcal{I}_{(a_1,\dots,a_{p-2},0,0)} \end{array} \right\} \\ & \cong \left\{ M \in k\mathbf{B}_{\rho,w}^\Sigma / I_{(a_1,\dots,a_{p-2},0)} - \text{mod} \mid \begin{array}{l} M \text{ has simple factors } D_\lambda \\ \lambda \in \mathcal{I}_{(a_1,\dots,a_{p-2},0,0)} \end{array} \right\}, \end{aligned}$$

where the latter isomorphism holds thanks to the quasi-heredity of the quotient $k\mathbf{B}_{\rho,w}^\Sigma / I_{\text{unres}}$. We deduce that, for $i = 1, \dots, p-2$,

$$k\mathbf{B}_{\rho,w}^\Sigma / I_{(a_1,\dots,a_{p-2},0)} - \text{mod} \cong$$

$$\{M \in k\Sigma_v - \text{mod} \mid M \text{ has composition factors } D^\lambda, \lambda \in \mathcal{I}_{(0,a_1,\dots,a_{p-2},0)} \cap \mathcal{I}_{\text{reg}}\}.$$

The second part of the lemma follows on an application of $\#$. \square

Note the following obvious fact:

Lemma 128 *Suppose that I, J are ideals in an algebra A . Then*

$$A/(I + J) - \text{mod} \cong (A/I - \text{mod}) \cap (A/J - \text{mod}). \square$$

Note that A_i and B_i both have simple modules in natural correspondence with the set $\{D^\lambda \mid \lambda \in \mathcal{K}_i\}$. Let $\mathcal{L}(i) = \mathcal{L}(0, \dots, 0, w, 0, \dots, 0)$ be a “single runner subcategory” of $k\Sigma_v - \text{mod}$, associated to runner i , for $i = 1, \dots, p-1$. Thus, $\mathcal{C}(i)$ is the Serre subcategory of $k\Sigma_v - \text{mod}$ generated by

$$\{D^\lambda \mid \lambda \in \mathcal{K}_i\}.$$

We may now produce “single runner quotients” of $k\mathbf{B}_{\rho,w}^\Sigma$, for $p \geq 3$.

Proposition 129 *Let $p \geq 3$.*

For $i = 1, \dots, p-1$, there exists an ideal α_i of $k\mathbf{B}_{\rho,w}^\Sigma$, such that

$$k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i - \text{mod} \cong \mathcal{L}(i).$$

There are natural algebra isomorphisms,

$$kA_i \cong kB_i \cong k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i.$$

Proof:

For ideals α_i , we take $I_i + J_i$, so long as $2 \leq i \leq p-2$. We take $\alpha_1 = I_1$, and $\alpha_{p-1} = J_{p-1}$. From lemmas 127 and 128 above it is clear that $k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i - \text{mod} \cong \mathcal{L}(i)$.

Let $i = 1, \dots, p-1$. We have $k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i = y_i (k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i) y_i$, since any projective summand of $k\mathbf{B}_{\rho,w}^\Sigma$, maximal subject to the restriction that its top lies in $\mathcal{C}(i)$, must be isomorphic to a summand of $k\Sigma_v y_i$.

Note that $\alpha_i = I_{i,q} + J_{i,q}(\text{mod } p)$, where $I_{i,q}$ (resp. $J_{i,q}$, $\alpha_{i,q}$) is the q -analogue of I_i (resp. J_i , α_i), an ideal of $\mathcal{O}\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$.

By proposition 126, and lemma 128, we know that $\mathcal{O}\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\alpha_{i,q} = KB_{i,q}$. We therefore witness the inclusion, $y_{i,q}I_{i,q}y_{i,q} \subseteq y_{i,q}J_{i,q}y_{i,q}$, over the field K . Taking intersections with the natural \mathcal{O} -form for $K\mathcal{H}_q(\Sigma_v)$, we reveal the inclusion $y_{i,q}I_{i,q}y_{i,q} \subseteq y_{i,q}J_{i,q}y_{i,q}$, over \mathcal{O} .

Reducing modulo p , we find that $y_i J_i y_i$ contains $y_i I_i y_i$. In conclusion,

$$k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i = y_i (k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i) y_i \cong y_i (k\mathbf{B}_{\rho,w}^\Sigma/J_i) y_i = kB_i,$$

for $i = 1, \dots, p-1$. Likewise, $k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i \cong kA_i$, for $i = 1, \dots, p-1$. \square

Generalising the above proposition and its proof, we have,

Proposition 130 *Let $p > 3$, and let $\sum_{i=2}^{p-2} a_i = w$.*

There exists an ideal $\mathcal{N}_{0,a_2,\dots,a_{p-2},0}$ of $k\mathbf{B}_{\rho,w}^\Sigma$ such that,

$$k\mathbf{B}_{\rho,w}^\Sigma/\mathcal{N}_{0,a_2,\dots,a_{p-2},0} - \text{mod} \cong \mathcal{L}(0, a_2, \dots, a_{p-2}, 0).$$

There are algebra isomorphisms,

$$kA_{(0,a_2,\dots,a_{p-2},0)} \cong kB_{(0,a_2,\dots,a_{p-2},0)} \cong k\mathbf{B}_{\rho,w}^\Sigma/\mathcal{N}_{0,a_2,\dots,a_{p-2},0}. \square$$

To complete this section, we bear an infant theorem 132.

Proposition 131 (a) Let $p = 2$. There is a nilpotent ideal α_1 of $k\mathbf{B}_{\rho,w}^\Sigma$, such that

$$k\mathbf{B}_{\rho,w}^\Sigma/\alpha_1 - \text{mod} \cong \mathcal{S}(w, w) - \text{mod}.$$

(b) Let $p \geq 3$, and let $i = 1, \dots, p-1$. There are equivalences of abelian categories,

$$\mathcal{L}(i) \cong k\mathbf{B}_{\rho,w}^\Sigma/\alpha_i - \text{mod} \cong \mathcal{S}(w, w) - \text{mod}.$$

Proof:

(a) This is theorem 90, in case $p = 2$.

(b) Let $i = p-1$. The first equivalence is then a particular case of proposition 129. To see the second equivalence recall from theorem 90, that there is a $k\mathbf{B}_{\rho,w}^\Sigma$ -module kM , with composition factors in $\mathcal{L}(p-1)$, such that $k\mathbf{B}_{\rho,w}^\Sigma/\text{Ann}(kM)$ is Morita equivalent to the Schur algebra $\mathcal{S}(w, w)$.

We have $\alpha_{p-1} \subseteq \text{Ann}(kM)$, by the first part of the proposition. A dimension count yields an isomorphism between $k\mathbf{B}_{\rho,w}^\Sigma/\text{Ann}(kM)$ and the quotient $k\mathbf{B}_{\rho,w}^\Sigma/\alpha_{p-1}$.

Let $i = 2, \dots, p-1$. Theorem 123 provides a Ringel duality between kA_i and kA_{i-1} . In the light of proposition 129, and the knowledge that $\mathcal{S}(w, w)$ is Ringel self-dual, we discover that $\mathcal{L}(i) \cong \mathcal{S}(w, w) - \text{mod}$, for $i = 1, \dots, p-1$. \square

A global-local theorem, and the James adjustment matrix of a Rock block.

In the final sections of this chapter, we prove the following:

Theorem 132 There is a Morita equivalence between $k\mathcal{G}b_{\rho,w}^\Sigma$ and a direct sum,

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right),$$

of tensor products of Schur algebras.

Under this Morita equivalence,

$$\mathcal{L}(a_1, \dots, a_{p-1}) \cong \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right) - \text{mod},$$

and the correspondence of simple modules is:

$$D^{[\emptyset, \dots, \lambda_{p-3}, \lambda_{p-2}, \lambda_{p-1}]} \leftrightarrow \dots \otimes L(\lambda'_{p-3}) \otimes L(\lambda_{p-2}) \otimes L(\lambda'_{p-1}).$$

The James adjustment matrix of $k\mathbf{B}_{\rho,w}^\Sigma$ is equal to the decomposition matrix of,

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right).$$

That is to write,

$$\begin{aligned} & [D_q^{[\emptyset, \dots, \lambda'_{p-3}, \lambda_{p-2}, \lambda'_{p-1}]} : D^{[\emptyset, \dots, \mu'_{p-3}, \mu_{p-2}, \mu'_{p-1}]}] = \\ & = \begin{cases} \prod_{i=1}^{p-1} [\Delta(\lambda_i) : L(\mu_i)], & \text{if } |\lambda_i| = |\mu_i|, i = 1, \dots, p-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Here, we set $\mathcal{S}(0, 0) \cong k$.

Remark 133 When $w < p$, the quotient $k\mathcal{G}b_{\rho,w}^\Sigma$ is merely the quotient of $k\mathbf{B}_{\rho,w}^\Sigma$ by its radical. We are therefore far from the strength of theorem 54.

Our proof of theorem 132 is inductive, on one hand applying the Ringel dualities of chapter 6, and on the other generalising the theory of chapter 5.

There falls an elegant description of the decomposition matrix of $k\mathbf{B}_{\rho,w}^\Sigma$, in terms of Littlewood-Richardson coefficients, and decomposition matrices of Schur algebras which are bounded in degree by w .

Corollary 134 The decomposition matrix of $k\mathbf{B}_{\rho,w}^\Sigma$ is equal to the matrix product,

$$\begin{aligned} & Dec_{<t-q>} \left(K[t]_{(t-q)} b_{\emptyset,1}^{\mathcal{H}_t} \wr \Sigma_w \right) \times \\ & Dec \left(\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i) \right) \right). \end{aligned}$$

Here, $Dec_{<t-q>} \left(K[t]_{(t-q)} b_{\emptyset,1}^{\mathcal{H}_t} \wr \Sigma_w \right)$ is the decomposition matrix of a wreath product of the principal block of $K[t]_{(t-q)} \mathcal{H}_t(\Sigma_p)$, with Σ_w . Formulae for the entries in this matrix are given in terms of Littlewood-Richardson coefficients (theorem 59). \square

Note that $kA_{(a_1, \dots, a_{p-1})}$ is the reduction modulo p of $\mathcal{O}A_{(a_1, \dots, a_{p-1}), q}$. The proof of theorem 132 rests upon the following proposition:

Proposition 135 *Let a_i be natural numbers such that $\sum_{i=1}^{p-1} a_i = w$. Then there exists an ideal $\mathcal{N}_{a_1, \dots, a_{p-1}}$ in $k\mathbf{B}_{\rho, w}^\Sigma$, such that,*

(a) $\mathcal{N}_{a_1, \dots, a_{p-1}}$ is equal to the reduction modulo p of $\mathcal{ON}_{a_1, \dots, a_{p-1}, q}$.

(b) There are isomorphisms,

$$k\mathbf{B}_{\rho, w}^\Sigma / \mathcal{N}_{a_1, \dots, a_{p-1}} \cong A_{(a_1, \dots, a_{p-1})} \cong B_{(a_1, \dots, a_{p-1})}.$$

(c) $k\mathbf{B}_{\rho, w}^\Sigma / \mathcal{N}_{a_1, \dots, a_{p-1}}$ is Morita equivalent to

$$\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i).$$

(d) Under the Morita equivalence of (3), the simple module

$$\dots \otimes L(\lambda'_{p-3}) \otimes L(\lambda_{p-2}) \otimes L(\lambda'_{p-1})$$

for $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$, corresponds to the simple $k\mathbf{B}_{\rho, w}^\Sigma$ -module $D^{[\emptyset, \dots, \lambda_{p-3}, \lambda_{p-2}, \lambda_{p-1}]}$ in $\mathcal{L}(a_1, \dots, a_{p-1})$.

(e) The decomposition matrix of the \mathcal{O} -algebra $A_{(a_1, \dots, a_{p-1})}$ is equal to the decomposition matrix of the quasi-hereditary algebra, $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$.

It is the concern of the last section of this chapter to prove proposition 135. First, let us give a proof of theorem 132 from proposition 135:

Let

$$\mathcal{N} = \bigcap_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \mathcal{N}_{a_1, \dots, a_{p-1}},$$

$$\Omega = \{(a_i)_{i=1, \dots, p-1} | a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{p-1} a_i = w\}.$$

Then, for $\alpha \in \Omega$, we know that $k\mathbf{B}_{\rho, w}^\Sigma / ((\bigcap_{\omega \in \Omega - \alpha} \mathcal{N}_\omega) + \mathcal{N}_\alpha) - \text{mod}$ is empty, by theorem 135(4) and lemma 128. We deduce that $k\mathbf{B}_{\rho, w}^\Sigma = ((\bigcap_{\omega \in \Omega - \alpha} \mathcal{N}_\omega) + \mathcal{N}_\alpha)$. By linear algebra, it follows that,

$$k\mathbf{B}_{\rho, w}^\Sigma / \mathcal{N} \cong \bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} k\mathbf{B}_{\rho, w}^\Sigma / \mathcal{N}_{a_1, \dots, a_{p-1}}.$$

Note that proposition 135(1) implies that \mathcal{N} is equal to the p -modular reduction of

$$\bigcap_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \mathcal{ON}_{a_1, \dots, a_{p-1}, q} = b_{\rho, w}^{\mathcal{H}_q} \cdot \mathcal{ON}_q.$$

Thus, $k\mathbf{B}_{\rho,w}^\Sigma/\mathcal{N}$ is isomorphic to $k\mathcal{G}b_{\rho,w}^\Sigma$.

These isomorphisms, along with proposition 135(3), complete the proof of the Morita equivalence of theorem 132, and the correspondence between simple modules under this Morita equivalence.

To see the correspondence between decomposition numbers, first note that the decomposition matrix of $\mathcal{G}b_{\rho,w}^\Sigma$ is equal to the decomposition matrix of

$$\bigoplus_{\substack{a_1, \dots, a_{p-1} \in \mathbb{Z}_{\geq 0} \\ \sum a_i = w}} \mathcal{O}A_{(a_1, \dots, a_{p-1}), q},$$

by proposition 126. Secondly, note that the decomposition matrix of $\mathcal{O}A_{(a_1, \dots, a_{p-1}), q}$ is equal to the decomposition matrix of $\mathcal{O}A_{(a_1, \dots, a_{p-1})}$, since both of these algebras are semisimple over K . Thirdly, note that the decomposition matrix of $\mathcal{O}A_{(a_1, \dots, a_{p-1})}$ is equal to the decomposition matrix of $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$, by proposition 135(5). \square

Induction.

The intent of this section is to convince the Reader of the truth of proposition 135. Let w be a natural number. We assume proposition 135 is proven for Rock blocks of weight strictly less than w , and deduce the same result for a Rock block $k\mathbf{B}_{\rho,w}^\Sigma$ of weight w .

Let a_{p-1} be a natural number, $0 < a_{p-1} \leq w$. Let $u = n - a_{p-1}p$.

Let $L_{a_{p-1}} = \Sigma_p^1 \times \dots \times \Sigma_p^{a_{p-1}} \times \Sigma_u^0 \leq \Sigma_v$, where $\Sigma_p^i = \text{Sym}\{(i-1)p + 1, \dots, ip\}$, and $\Sigma_u^0 = \text{Sym}\{a_{p-1}p + 1, \dots, wp + r\}$.

Let $e_{a_{p-1}}$ be an idempotent of $k\Sigma_v$, defined to be the product of block idempotents of L_i with cores $\emptyset, \dots, \emptyset, \rho$, for $i = 0, \dots, a_{p-1}$.

Let $\zeta_{a_{p-1}} = \sum_{x \in \Sigma_p^1 \times \dots \times \Sigma_p^{a_{p-1}}} x$. Let $kM_{a_{p-1}} = \mathcal{O}\Sigma_v e_{a_{p-1}} \zeta_{a_{p-1}}$.

Suppose that a_1, \dots, a_{p-1} are natural numbers, whose sum is w . By our inductive assumption, there is an ideal $\mathcal{N}_{a_1, \dots, a_{p-2}, 0}$ of $k\mathbf{B}_{\rho, w-a_{p-1}}^\Sigma$, such that $k\mathbf{B}_{\rho, w-a_{p-1}}^\Sigma/\mathcal{N}_{a_1, \dots, a_{p-2}, 0}$ is Morita equivalent to $\bigotimes_{i=1}^{p-2} \mathcal{S}(a_i, a_i)$, and whose simple modules are the simple objects of $\mathcal{L}(a_1, \dots, a_{p-2}, 0)$.

Let $kM_{(a_1, \dots, a_{p-1})}$ be equal to the quotient $(kM_{a_{p-1}}/kM_{a_{p-1}}\mathcal{N}_{a_1, \dots, a_{p-2}, 0})$. Since $kM_{a_{p-1}}$ is projective as a right $k\Sigma_u^0$ -module, we have

$$kM_{(a_1, \dots, a_{p-1})} \cong kM_{a_{p-1}} \bigotimes_{\Sigma_u^0} \left(k\mathbf{B}_{\rho, w-a_{p-1}}^\Sigma / \mathcal{N}_{a_1, \dots, a_{p-2}, 0} \right).$$

Therefore $kM_{(a_1, \dots, a_{p-1})}$ is a $k\Sigma_v$ - $k\Sigma_u^0$ -bimodule.

The following proposition, and its proof, generalise theorem 90.

Proposition 136 (a) Consider the $k\mathbf{B}_{\rho,w}^\Sigma$ -module $kM_{(a_1,\dots,a_{p-1})}$. Its endomorphism ring $k\mathcal{E}_{(a_1,\dots,a_{p-1})}$ is isomorphic to $k\mathbf{B}_{\rho,w-a_{p-1}}^\Sigma/\mathcal{N}_{a_1,\dots,a_{p-2},0} \otimes k\Sigma_{a_{p-1}}$.

(b) The quotient of $k\mathbf{B}_{\rho,w}^\Sigma$, by the annihilator $kI_{(a_1,\dots,a_{p-1})}$ of $kM_{(a_1,\dots,a_{p-1})}$, is Morita equivalent to the tensor product,

$$\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i),$$

of Schur algebras.

(c) The $\left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)\right) - \left(\bigotimes_{i=1}^{p-2} \mathcal{S}(a_i, a_i)\right) \otimes k\Sigma_{a_{p-1}}$ - bimodule which corresponds via Morita equivalence to the $k\mathbf{B}_{\rho,w}^\Sigma/kI_{(a_1,\dots,a_{p-1})}$ - $k\mathcal{E}_{(a_1,\dots,a_{p-1})}$ bimodule $kM_{(a_1,\dots,a_{p-1})}$ is isomorphic to,

$$\left(\bigotimes_{i=1}^{p-2} \mathcal{S}(a_i, a_i)\right) \otimes E^{\otimes a_{p-1}\#}.$$

(d) Under the Morita equivalence between $k\mathbf{B}_{\rho,w}^\Sigma/kI_{(a_1,\dots,a_{p-1})}$ and the tensor product $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$, the correspondence between simple modules is:

$$D^{[\emptyset, \dots, \lambda_{p-3}, \lambda_{p-2}, \lambda_{p-1}]} \leftrightarrow \dots \otimes L(\lambda'_{p-3}) \otimes L(\lambda_{p-2}) \otimes L(\lambda'_{p-1}). \square$$

Let $KM_{(a_1,\dots,a_{p-1}),q}$ be the q -analogue of $kM_{(a_1,\dots,a_{p-1})}$.

The proposition below is a q -analogue of proposition 90, valid in characteristic zero.

Proposition 137 Let K be a splitting field for $\mathcal{H}_q(\Sigma_v)$. Let a_1, \dots, a_{p-1} be natural numbers whose sum is w .

(a) $KA_{(a_1,\dots,a_{p-1}),q}$ is a semisimple algebra.

(b) $KM_{(a_1,\dots,a_{p-2},a_{p-1}),q}$ is a semisimple $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ -module. Its endomorphism ring $K\mathcal{E}_{(a_1,\dots,a_{p-1}),q}$ is isomorphic to $\left(K\mathbf{B}_{\rho,w-a_{p-1}}^{\mathcal{H}_q}/K\mathcal{N}_{a_1,\dots,a_{p-2},0,q}\right) \otimes K\Sigma_{a_{p-1}}$.

(c) The quotient of $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$, by the annihilator $KI_{(a_1,\dots,a_{p-1}),q}$ of $M_{(a_1,\dots,a_{p-1}),q}$ is Morita equivalent to the tensor product, $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$ of semisimple Schur algebras.

(d) The $\left(\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)\right) - \left(\bigotimes_{i=1}^{p-2} \mathcal{S}(a_i, a_i)\right) \otimes K\Sigma_{a_{p-1}}$ -bimodule, which corresponds (under Morita equivalence) to $KM_{(a_1, \dots, a_{p-1}), q}$ is isomorphic to

$$\left(\bigotimes_{i=1}^{p-2} \mathcal{S}(a_i, a_i)\right) \otimes E^{\otimes a_{p-1} \#}.$$

(e) The annihilator $KI_{(a_1, \dots, a_{p-1}), q}$ is precisely $\mathcal{N}_{a_1, \dots, a_{p-1}, q}$.

Proof:

Dirichlet's theorem guarantees the existence of infinitely many prime numbers l' , such that $l' \equiv 1 \pmod{p}$ (i.e. such that $\mathbb{F}_{l'}$ contains primitive p^{th} roots of unity). Let us choose such a prime, such that $w < l'$.

Let q be a primitive p^{th} root of unity. A second application of Dirichlet's theorem provides a prime number q' , such that $q' \equiv q \pmod{l'}$.

Let $(K_{l'}, \mathcal{O}_{l'}, k_{l'})$ be an l' -modular system, such that $K_{l'}$ is a splitting field for $GL_v(q')$, and such that $q \in \mathcal{O}_{l'}$.

By theorem 80, there is an equivalence between $k_{l'} b_{\rho, w}^{\mathcal{H}_q}$, and $k_{l'} b_{\emptyset, 1}^{\mathcal{H}_q} \wr \Sigma_w$. The decomposition matrix of this algebra is equal to the decomposition matrix of $K_{l'} b_{\rho, w}^{\mathcal{H}_q}$, by theorem 59. Therefore, the James adjustment algebra $K\mathcal{G}b_{\rho, w}^{\mathcal{H}_q}$ is semisimple, over ANY splitting field K .

The proposition is now visible, by induction on w . \square

Here goes the induction. *Proof of proposition 135:*

How to define the ideals $\mathcal{N}_{a_1, \dots, a_{p-1}}$?

Case 1: if $a_1 = a_{p-1} = 0$, the ideal of proposition 130 suffices.

Case 2: If $a_{p-1} \neq 0$, proposition 90 above provides the ideal : set $\mathcal{N}_{a_1, \dots, a_{p-1}} = kI_{(a_1, \dots, a_{p-1})}$.

Case 3: If $a_{p-1} = 0$, and $a_1 \neq 0$, set $\mathcal{N}_{a_1, \dots, a_{p-1}} = \mathcal{N}_{a_{p-1}, \dots, a_1}^{\#}$.

(a) Case 1: Note that, by a q -analogue of proposition 130, we know that $I_{(0, a_2, \dots, a_{p-2}, 0), q} + J_{(0, a_2, \dots, a_{p-2}, 0), q}$ is contained in $\mathcal{N}_{0, a_2, \dots, a_{p-2}, 0, q}$, over the field K , and hence also over \mathcal{O} . Thus,

$$\mathcal{N}_{0, a_2, \dots, a_{p-2}, 0} \subseteq \mathcal{N}_{0, a_2, \dots, a_{p-2}, 0, q} \pmod{p}.$$

However, proposition 126 and proposition 130 imply that both quotients, $K\mathbf{B}_{\rho, w}^{\mathcal{H}_q} / \mathcal{N}_{0, a_2, \dots, a_{p-2}, 0, q}$ and $k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{0, a_2, \dots, a_{p-2}, 0}$ have the same dimension, equal to the dimension of $A_{(a_1, \dots, a_{p-1})}$.

Case 2: Let $KM_{(a_1, \dots, a_{p-1}), q}$ be the q -analogue of $kM_{(a_1, \dots, a_{p-1})}$. By proposition 137, $KM_{(a_1, \dots, a_{p-1}), q}$ is a semisimple $K\mathcal{H}_q(\Sigma_v)$ -module, and the quotient $\mathcal{OB}_{\rho, w}^{\mathcal{H}_q} / \mathcal{ON}_{a_1, \dots, a_{p-1}, q}$ surjects onto $k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{a_1, \dots, a_{p-1}}$.

Furthermore, on writing $k\mathcal{E}_{(a_1, \dots, a_{p-1})}$ (respectively $K\mathcal{E}_{(a_1, \dots, a_{p-1}), q}$) for the endomorphism ring of $kM_{(a_1, \dots, a_{p-1})}$ (respectively $KM_{(a_1, \dots, a_{p-1}), q}$), we have

$$k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{a_1, \dots, a_{p-1}} = \text{End}_{k\mathcal{E}_{(a_1, \dots, a_{p-1})}}(kM_{(a_1, \dots, a_{p-1})}),$$

$$K\mathbf{B}_{\rho, w}^{\mathcal{H}_q} / K\mathcal{N}_{a_1, \dots, a_{p-1}, q} = \text{End}_{K\mathcal{E}_{(a_1, \dots, a_{p-1}), q}}(KM_{(a_1, \dots, a_{p-1}), q}).$$

By propositions 136 and 137, these two endomorphism rings have the same dimensions. Thus, $\mathcal{N}_{a_1, \dots, a_{p-1}} = \mathcal{ON}_{a_1, \dots, a_{p-1}, q}(\text{modulo } p)$.

Case 3: Note that

$$\mathcal{N}_{a_1, \dots, a_{p-1}} = \mathcal{N}_{a_{p-1}, \dots, a_1}^{\#} =$$

$$\mathcal{N}_{a_{p-1}, \dots, a_1, q}^{\#}(\text{modulo } p) = \mathcal{N}_{a_1, \dots, a_{p-1}, q}(\text{modulo } p).$$

(b) follows from proposition 126 and (a), by p -modular reduction.

(c) What is the Morita type of the quotient $k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{a_1, \dots, a_{p-1}}$?

In case 2, $a_{p-1} \neq 0$, we know that $k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{a_1, \dots, a_{p-1}}$ is Morita equivalent to $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$, by proposition 136. Recall that $\mathcal{S}(n, n)$ is Ringel self-dual for any n (theorem 17). The Ringel dualities of theorem 123, and the isomorphisms of part (b), show that $k\mathbf{B}_{\rho, w}^{\Sigma} / \mathcal{N}_{a_1, \dots, a_{p-1}}$ is Morita equivalent to $\bigotimes_{i=1}^{p-1} \mathcal{S}(a_i, a_i)$ in general.

(d) Tracing back through proposition 136, along remark 124, past theorem 123 to proposition 61, the correspondence between simple modules is visible.

(e) The decomposition matrix of $A_{(a_1, \dots, a_{p-1})}$ as an \mathcal{O} -algebra is equal to the decomposition matrix $([\Delta(\lambda) : L(\mu)])$ of $kA_{(a_1, \dots, a_{p-1})}$ as a quasi-hereditary algebra. (e) is now clear from (b) and (c). \square

Chapter VIII

Doubles, Schur super-bialgebras, and Rock blocks of Hecke algebras.

J. Alperin's weight conjecture [1], now along with a number of examples (eg. theorem 39, theorem 132, [22]), suggests that the category of modules for a finite group algebra, over a field of prime characteristic, should resemble a highest weight category. However, group algebras are symmetric algebras, and therefore far from quasi-hereditary. This chapter presents a conjectural resolution to this problem, for symmetric groups.

Indeed, we associate symmetric associative algebras to certain bialgebras, via a double construction (theorem 138). To any super-algebra, we then assign a "Schur super-bialgebra". From the algebra of $n \times n$ matrices, concentrated in parity zero, we thus recover the classical Schur bialgebra, $\mathcal{S}(n)$. Applying the aforementioned double construction to certain Schur super-bialgebras, which correspond to quivers of type A , we reveal symmetric algebras which should be Morita equivalent to Rock blocks for Hecke algebras (conjecture 165).

A double construction.

Let k be a field. Let B be an bialgebra over k , endowed with a k -endomorphism σ , which is an algebra anti-homomorphism, and a coalgebra anti-homomorphism. Suppose that B is graded, with finite dimensional graded pieces. Let B^* be the graded dual of B . Then B^* is a bialgebra, whose product is dual to the coproduct on B , and whose coproduct is dual to the product on B .

Let us write comultiplication as $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Theorem 138 *The tensor product $D(B) = B \otimes B^*$ is a k -algebra, with associative product given by,*

$$(a \otimes \alpha).(b \otimes \beta) = \sum a_{(2)} b_{(1)} \otimes \beta_{(2)} \alpha_{(1)} \langle a_{(1)}^\sigma, \beta_{(1)} \rangle \langle \alpha_{(2)}, b_{(2)}^\sigma \rangle .$$

Furthermore, $D(B)$ possesses a symmetric associative bilinear form,

$$\langle a \otimes \alpha, b \otimes \beta \rangle = \langle a^\sigma, \beta \rangle \langle \alpha, b^\sigma \rangle .$$

Therefore, if σ is invertible, then $D(B)$ is a symmetric algebra.

So long as B is cocommutative, there are algebra homomorphisms,

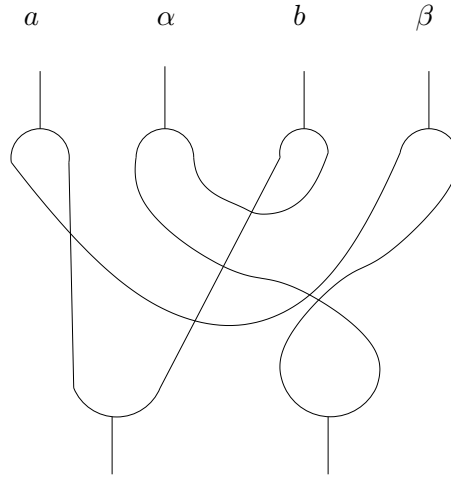
$$\Delta_l : D(B) \rightarrow D(B) \otimes B,$$

$$\Delta_l : a \otimes \alpha \mapsto \sum a_{(1)} \otimes \alpha \otimes a_{(2)},$$

$$\Delta_r : D(B) \rightarrow B \otimes D(B)$$

$$\Delta_r : a \otimes \alpha \mapsto \sum a_{(1)} \otimes a_{(2)} \otimes \alpha.$$

Beneath is a picture of the product $a \otimes \alpha$ and $b \otimes \beta$ in $D(B)$. We discovered this product, upon studying the group algebra of the principal block of Σ_5 , in characteristic two.



Proof:

We first check associativity:

$$\begin{aligned} & ((a \otimes \alpha).(b \otimes \beta)).(c \otimes \gamma) \\ &= (\sum a_{(2)} b_{(1)} \otimes \beta_{(2)} \alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma >).(c \otimes \gamma) \\ &= \sum (a_{(2)} b_{(1)})_{(2)} c_{(1)} \otimes (\beta_{(2)} \alpha_{(1)})_{(1)} \gamma_{(2)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma > \\ &\quad < (a_{(2)} b_{(1)})_{(1)}^\sigma, \gamma_{(1)} > < (\beta_{(2)} \alpha_{(1)})_{(2)}, c_{(2)}^\sigma > \\ &= \sum a_{(3)} b_{(2)} c_{(1)} \otimes \gamma_{(2)} \beta_{(2)} \alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(3)}, b_{(3)}^\sigma > \\ &\quad < b_{(1)}^\sigma a_{(2)}^\sigma, \gamma_{(1)} > < \beta_{(3)} \alpha_{(2)}, c_{(2)}^\sigma > \\ &= \sum a_{(3)} b_{(2)} c_{(1)} \otimes \gamma_{(2)} \beta_{(2)} \alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(3)}, b_{(3)}^\sigma > \\ &\quad < b_{(1)}^\sigma, \gamma_{(1)(1)} > < a_{(2)}^\sigma, \gamma_{(1)(2)} > < \alpha_{(2)}, c_{(2)(1)}^\sigma > < \beta_{(3)}, c_{(2)(2)}^\sigma > \\ &= \sum a_{(3)} b_{(2)} c_{(1)} \otimes \gamma_{(3)} \beta_{(2)} \alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(3)}, b_{(3)}^\sigma > \\ &\quad < b_{(1)}^\sigma, \gamma_{(1)} > < a_{(2)}^\sigma, \gamma_{(2)} > < \alpha_{(2)}, c_{(2)}^\sigma > < \beta_{(3)}, c_{(3)}^\sigma >. \end{aligned}$$

This final symmetric expression may similarly be shown to equal $(a \otimes \alpha).((b \otimes \beta).(c \otimes \gamma))$. Associativity is proven !

Now suppose that σ is invertible. There can be little doubt of the symmetry, nor the non-degeneracy of the bilinear form we have defined on $D(B)$. What about associativity ?

$$\begin{aligned}
& < (a \otimes \alpha).(b \otimes \beta), (c \otimes \gamma) > \\
&= \sum < a_{(2)}b_{(1)} \otimes \beta_{(2)}\alpha_{(1)}, c \otimes \gamma > < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma > \\
&= \sum < b_{(1)}^\sigma a_{(2)}^\sigma, \gamma > < \beta_{(2)}\alpha_{(1)}, c^\sigma > < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma > \\
&= \sum < b_{(1)}^\sigma, \gamma_{(1)} > < a_{(2)}^\sigma, \gamma_{(2)} > < \alpha_{(1)}, c_{(1)}^\sigma > \\
&\quad < \beta_{(2)}, c_{(2)}^\sigma > < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma >.
\end{aligned}$$

This final symmetric expression may similarly be shown to equal $< (a \otimes \alpha), (b \otimes \beta).(c \otimes \gamma) >$. Associativity of $<, >$ is proven !

The last check we make is that Δ_r is an algebra homomorphism, so long as B is cocommutative (a similar calculation can be written down for Δ_l):

$$\begin{aligned}
& \Delta_r((a \otimes \alpha).(b \otimes \beta)) \\
&= \sum (a_{(2)}b_{(1)})_{(1)} \otimes (a_{(2)}b_{(1)})_{(2)} \otimes \beta_{(2)}\alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma > \\
&= \sum a_{(2)}b_{(1)} \otimes a_{(3)}b_{(2)} \otimes \beta_{(2)}\alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(3)}^\sigma > \\
&= \sum a_{(1)}b_{(1)} \otimes a_{(3)}b_{(2)} \otimes \beta_{(2)}\alpha_{(1)} < a_{(2)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(3)}^\sigma > \\
&= \sum a_{(1)}b_{(1)} \otimes a_{(2)(2)}b_{(2)(1)} \otimes \beta_{(2)}\alpha_{(1)} < a_{(2)(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)(2)}^\sigma > \\
&= (a_{(1)} \otimes a_{(2)} \otimes \alpha).(b_{(1)} \otimes 1 \otimes b_{(2)} \otimes \beta) \\
&= \Delta_r(a \otimes \alpha).\Delta_r(b \otimes \beta). \quad \square
\end{aligned}$$

Remark 139 Since B possesses an algebra anti-automorphism σ , the dual of a left/right B -module may be given the structure of a left/right B -module as well. The left/right regular action of B on itself, may thus be dualised to define a left/right action of B on B^* . We obtain a simpler expression,

$$(a \otimes \alpha).(b \otimes \beta) = \sum a_{(2)}b_{(1)} \otimes (a_{(1)} \circ \beta)(\alpha \circ b_{(2)}),$$

for the associative product on $D(B)$.

Remark 140 When B is cocommutative, Δ_l and Δ_r both give $D(B)$ the structure of a B -comodule. The coproducts Δ_l and Δ_r satisfy the following property:

Let M be a B -module, and let N be a $D(B)$ -module. The $D(B)$ -module $M \otimes N$ (formed via Δ_r) is isomorphic to the $D(B)$ -module $N \otimes M$ (formed via Δ_l).

In the examples of this article, we find ourselves in the situation of the following lemma. Its proof is a routine check.

Lemma 141 *Suppose that $B = \bigoplus_{r \in \mathbb{Z}_+} B(r)$ is a bialgebra, which is direct sum of finite dimensional pieces, $B(r)$. Suppose that B possesses a degree preserving k -endomorphism σ , which is an algebra antiautomorphism, and a coalgebra automorphism. We write B^* for the graded dual, $\bigoplus_{r \in \mathbb{Z}_+} B(r)^*$, of B . Suppose further, that*

- i. $B(r)$ is a subalgebra of B , for $r \in \mathbb{Z}_+$.*
- ii. $B(0) \text{congk}$, and the projection and embedding maps between $B(0)$ and B give B the structure of an augmented coalgebra.*
- iii. $B = \bigoplus_{r \in \mathbb{Z}_+} B(r)$ is a graded coalgebra. Thus,*

$$\Delta : B(r) \rightarrow \bigoplus_{d=0}^r B(r-d) \otimes B(d).$$

Then the degree r part of $D = D(B)$,

$$D(r) = \bigoplus_{d=0}^r B(r-d) \otimes B^*(d),$$

is a finite-dimensional, graded, symmetric algebra summand of D , where $B(r-d) \otimes B^(d)$ is given degree d .*

The ideal,

$$\mathcal{N}(r) = \bigoplus_{d=1}^r B(r-d) \otimes B^*(d),$$

of $D(r)$ is nilpotent.

The quotient $D(r)/\mathcal{N}(r)$ is isomorphic to the degree zero part $D^0(r) = B(r)$, of $D(r)$.

Irreducible $D(r)$ -modules are in natural correspondence with irreducible $B(r)$ -modules.

In this way, D is a graded associative algebra, whose degree zero part is isomorphic to B , as an algebra.

Upon writing \mathcal{N} for the ideal $\bigoplus_{r \in \mathbb{Z}_+} \mathcal{N}(r)$ of D , a splitting of the natural algebra monomorphism $B \rightarrow D$ becomes visible:

$$D \rightarrow D/\mathcal{N} \cong B.$$

The degree d part $B(r-d) \otimes B^(d)$ of $D(r)$ inherits a $D^0(r)$ - $D^0(r)$ -bimodule structure from $D(r)$, for $d = 0, \dots, r$. This is nothing but the natural $B(r)$ - $B(r)$ -bimodule structure on $B(r-d) \otimes B^*(d)$. \square*

We require a super- generalisation of theorem 138.

Suppose that B is a super-bialgebra. Thus, B is $\mathbb{Z}/2$ -graded algebra and coalgebra, so that the product m , and the coproduct Δ , preserve the grading:

$$m : B^i \otimes B^j \rightarrow B^{i+j},$$

$$\Delta : B^k \rightarrow \bigoplus_{i+j=k} B^i \otimes B^j, \text{ for } i, j, k \in \mathbb{Z}/2.$$

In addition,

$$\Delta(a.b) = \sum (-1)^{|a_{(2)}||b_{(1)}|} a_{(1)}.b_{(1)} \otimes a_{(2)}.b_{(2)}.$$

Suppose that B is endowed with a parity preserving endomorphism σ , which is a coalgebra anti-automorphism, and an algebra anti-automorphism.

Theorem 142 *The tensor product $D(B) = B \otimes B^*$ is a super-algebra, with associative product given by,*

$$(a \otimes \alpha).(b \otimes \beta) =$$

$$\sum (-1)^{s(a, \alpha, b, \beta)} a_{(2)}b_{(1)} \otimes \beta_{(2)}\alpha_{(1)} < a_{(1)}^\sigma, \beta_{(1)} > < \alpha_{(2)}, b_{(2)}^\sigma >,$$

where

$$s(a, \alpha, b, \beta) = |a_{(1)}|(|a_{(2)}| + |b_{(1)}|) + |b_{(1)}||\alpha| + |\alpha_{(1)}||\beta|,$$

and $\mathbb{Z}/2$ -grading, given by

$$|a \otimes \alpha| = |a| + |\alpha|.$$

In fact, $D(B)$ is endowed with a symmetric associative bilinear form,

$$< a \otimes \alpha, b \otimes \beta > = < a^\sigma, \beta > < \alpha, b^\sigma > .$$

Therefore, if σ is invertible, $D(B)$ is a symmetric super-algebra.

Proof:

Write out the proof of theorem 138 diagrammatically, rather than algebraically (thus, a variable should be represented by a string, a product by the fraying of a string into two parts, a coproduct by the joining of two strings together, etc.). To generalise this proof to the super- situation, we need only introduce the sign $(-1)^{|a||b|}$ whenever two strings (corresponding to variables a, b , in degrees $|a|, |b|$) cross.

The sign allocated to our product diagram is -1 , raised to the power

$$|a_{(1)}|(|a_{(2)}| + |b_{(1)}|) + |b_{(1)}|(|\alpha_{(1)}| + |\alpha_{(2)}|) + |\alpha_{(1)}|(|\beta_{(1)}| + |\beta_{(2)}|).$$

A slightly simpler expression is $(-1)^{s(a, \alpha, b, \beta)}$. \square

Remark 143 The \mathbb{Z}_+ -grading on B of lemma 141 is necessarily a different grading from the $\mathbb{Z}/2$ -grading on B of theorem 142. Indeed, lemma 141 generalises to apply to a bialgebra which is $\mathbb{Z}_+ \times \mathbb{Z}/2$ -graded, so that the \mathbb{Z} -grading is compatible with lemma 141, whilst the $\mathbb{Z}/2$ -grading is compatible with theorem 142.

Examples.

Example 144 Let $B = \mathcal{S}(1)$ be the Schur bialgebra associated to $GL_1(k)$, with trivial antiautomorphism. As a coalgebra, $\mathcal{S}(1)$ is the graded dual of the polynomial ring $k[X]$ in one variable. Each homogeneous component $\mathcal{S}(1, r)$ is isomorphic (as an algebra) to a copy of the field k .

Then we are in the context of lemma 141, and $D(r)$ is isomorphic to the uniserial algebra $k[Y]/(Y^r)$. \square

Example 145 Let $B = B(0) \oplus B(1) = k \oplus T_n(k)$, be the direct sum of a copy of the field k (in degree zero), and the algebra of $n \times n$ upper triangular matrices (in degree one). On writing ϵ for the unit in $B(0)$ (*not a unit for B*), we see that B is a cocommutative bialgebra via the coproduct,

$$\Delta : x \mapsto x \otimes \epsilon + \epsilon \otimes x, \quad x \in B(1)$$

$$\Delta : \epsilon \mapsto \epsilon \otimes \epsilon.$$

The bialgebra B possesses an algebra anti-automorphism σ , acting trivially on $B(0)$, but non-trivially on $B(1)$ taking $E_{i,j}$ to $E_{n-j+1, n-i+1}$. This map σ is a coalgebra anti-automorphism.

We are in the setup of lemma 141, and $D(1)$ is isomorphic to the path algebra of the circular quiver with n vertices, and clockwise orientation, modulo the ideal of paths of length $\geq n + 1$. This is a uniserial algebra, otherwise known as the Brauer tree algebra of a star, with multiplicity one ([2], chapter 5). \square

Example 146 Let Q be a quiver, without loops or multiple edges, equipped with an orientation reversing automorphism. Let $B = k \oplus kQ/I_2$ be the direct sum of a copy of the field k (in degree zero), and the path algebra kQ , modulo the ideal of paths of length ≥ 2 (in degree one). Just as in example 145, B may be given the structure of a bialgebra, equipped with an algebra anti-automorphism which is a coalgebra anti-automorphism.

This time, $D(1)$ is isomorphic to the zigzag algebra (see [43]), whose graph is the underlying graph of Q . If the underlying graph is an ordinary

Dynkin diagram of type A , the zigzag algebra is otherwise known as a linear Brauer tree algebra, with multiplicity one. \square

Brauer tree algebras appear naturally in the block theory of finite groups with cyclic defect groups. There appear to be more mysterious instances of doubles appearing in finite group theory:

Example 147 Let k be a field of characteristic two. Let $B = \mathcal{S}(2)$ be the Schur bialgebra associated to $GL_2(k)$, with transpose antiautomorphism. Then $D(2)$ is Morita equivalent to the Rock block $k\mathbf{B}_{\rho,2}^\Sigma$, otherwise known as the principal block of $k\Sigma_5$. This follows from Erdmann's description of the basic algebras for tame blocks of group algebras [29]. \square

Example 148 Let k be a field of characteristic two. Let $B = \mathcal{S}(2)$ be the Schur bialgebra associated to $GL_2(k)$, with transpose antiautomorphism. Let

$$C = (0 \rightarrow D(2)\xi_{(2)} \otimes \xi_{(2)}D(2)\xi_{(1^2)} \rightarrow D(2)\xi_{(1^2)} \rightarrow 0).$$

Here, the differential is given by the product map in $D(2)$. Then C is a tilting complex for $D(2)$, and its endomorphism ring \mathcal{E} in the homotopy category is Morita equivalent to $k\Sigma_4$. This follows from Holm's description [42] of derived equivalences between tame blocks of group algebras. \square

Remark 149 The equivalences of examples 147 and 148 both lift to Hecke algebras at -1 , over fields of arbitrary characteristic.

Notation.

Let V be a vector space.

We write $\bigwedge(V)$ for the exterior algebra on V , the coinvariants of the signature action of $\times_{r \geq 0} \Sigma_r$ on the tensor algebra $T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$. If v_1, \dots, v_n is a basis for V , then $\{v_{i_1} \wedge \dots \wedge v_{i_r} \mid i_1 < \dots < i_r, r \geq 0\}$ is a basis for $\bigwedge(V)$, where $v_{i_1} \wedge \dots \wedge v_{i_r}$ is the image in $\bigwedge(V)$ of $v_{i_1} \otimes \dots \otimes v_{i_r}$.

We write $\bigvee(V)$ for the invariants of the signature action of $\times_{r \geq 0} \Sigma_r$ on $T(V)$. If v_1, \dots, v_n is a basis for V , then $\{v_{i_1} \vee \dots \vee v_{i_r} \mid i_1 < \dots < i_r, r \geq 0\}$ is a basis for $\bigvee(V)$, where $v_{i_1} \vee \dots \vee v_{i_r}$ is the anti-symmetrisation of $v_{i_1} \otimes \dots \otimes v_{i_r}$.

We write $\mathcal{A}(V)$ for the symmetric algebra on V , the coinvariants of the permutation action of $\times_{r \geq 0} \Sigma_r$ on the tensor algebra $T(V) = \bigoplus_{r \geq 0} V^{\otimes r}$. If v_1, \dots, v_n is a basis for V , then $\{v_{i_1} \dots v_{i_r} \mid i_1 \leq \dots \leq i_r, r \geq 0\}$ is a basis for $\mathcal{A}(V)$, where $v_{i_1} \dots v_{i_r}$ is the image in $\mathcal{A}(V)$ of $v_{i_1} \otimes \dots \otimes v_{i_r}$.

We write $\mathcal{S}(V)$ for the invariants of the permutation action of $\times_{r \geq 0} \Sigma_r$ on $T(V)$. If v_1, \dots, v_n is a basis for V , then $\{v_{i_1} * \dots * v_{i_r} | i_1 \leq \dots \leq i_r, r \geq 0\}$ is a basis for $\mathcal{S}(V)$, where $v_{i_1} \dots v_{i_r}$ is the symmetrisation of $v_{i_1} \otimes \dots \otimes v_{i_r}$.

Now suppose M is the algebra of $n \times n$ matrices. Then we write $\bigwedge(n) = \bigwedge(M^*)$, $\bigvee(n) = \bigvee(M)$, and $\mathcal{A}(n) = \mathcal{A}(M^*)$, $\mathcal{S}(n) = \mathcal{S}(M)$. We write $\bigwedge(n, r)$, $\bigvee(n, r)$, $\mathcal{A}(n, r)$, $\mathcal{S}(n, r)$ for the r^{th} homogeneous components of these various spaces.

Schur super-bialgebras.

Let A, B be super-algebras (i.e. $\mathbb{Z}/2$ -graded associative algebras). Their tensor product, $A \otimes B$, becomes a super-algebra, with parity,

$$|a \otimes b| = |a| + |b|,$$

and super-product,

$$(a \otimes b).(a' \otimes b') = (-1)^{|b||a'|}(aa' \otimes bb').$$

For a super-algebra A , we define the super-algebra $A^{\otimes r}$ inductively, to be $A^{\otimes r} = A^{\otimes r-1} \otimes A$, with super-product as above.

The symmetric group Σ_r acts naturally as parity-preserving automorphisms on $A^{\otimes r}$. A simple reflection $(i, i+1) \in \Sigma_r$ acts as:

$$(a_1 \otimes \dots \otimes a_r)^{(i, i+1)} = (-1)^{|a_i||a_{i+1}|} a_1 \otimes \dots \otimes a_{i-1} \otimes a_{i+1} \otimes a_i \otimes a_{i+2} \otimes \dots \otimes a_r.$$

Let A be a super-algebra. Let $T(A)$ be the direct sum,

$$T(A) = \bigoplus_{d \geq 0} A^{\otimes d},$$

of super-algebras. This algebra becomes a super-bialgebra, with parity,

$$|a_1 \otimes \dots \otimes a_r| = |a_1| + \dots + |a_r|,$$

and coassociative comultiplication,

$$\Delta(a_1 \otimes \dots \otimes a_r) = \sum_{i=0}^r (a_1 \otimes \dots \otimes a_i) \otimes (a_{i+1} \otimes \dots \otimes a_r).$$

Definition 150 *Let A be a super-algebra. The Schur super-bialgebra associated to A is the graded sub-super-bialgebra,*

$$\mathcal{S}(A) = \bigoplus_{r \geq 0} \mathcal{S}(A)(r) = \bigoplus_{r \geq 0} (A^{\otimes r})^{\Sigma_r},$$

of Σ_r -fixpoints on $T(A)$.

Remark 151 In case $A = M = M_n(k)$ is the algebra of $n \times n$ matrices, concentrated in parity zero, we recover the classical Schur bialgebra $\mathcal{S}(n)$ in this way.

Remark 152 If A is a super-algebra, equipped with a parity-preserving anti-automorphism σ , then $A^{\otimes r}$ may also be equipped with a parity-preserving anti-automorphism,

$$\sigma : a_1 \otimes \dots \otimes a_r \mapsto \sigma(a_r) \otimes \dots \otimes \sigma(a_1).$$

Indeed, this map acts as a coalgebra anti-automorphism on $T(A)$, and restricts to a coalgebra anti-automorphism on $\mathcal{S}(A)$.

Therefore, if A is a super-algebra, equipped with a parity-preserving anti-automorphism σ , then $\mathcal{S}(A)$ is a super-bialgebra, equipped with an endomorphism σ , which is an algebra and coalgebra anti-automorphism. Under such circumstances, we may apply theorem 142, and form the double, $D(\mathcal{S}(A))$.

Schiver super-bialgebras.

The example which concerns us most in this booklet, is the special case when A is Morita equivalent to the path algebra of a quiver Q , modulo the ideal of paths of length > 1 . We are particularly interested in this when Q is a Dynkin quiver of type A .

Let Q be a quiver (that is, a locally finite, oriented graph). We note its vertex set V , and its set of edges E . For any edge $e \in E$, we denote its source $s(e) \in V$, and its tail $t(e) \in V$.

Let P_Q be the path algebra of Q , modulo the ideal of paths of length > 1 . To any natural number n , let us assign the algebra $P_Q(n)$, which is Morita equivalent to P_Q , and whose simple modules all have dimension n . Thus, $P_Q(n) = \text{End}_{P_Q}(P_Q^{\oplus n})$, and as vector spaces we have

$$P_Q(n) \cong M^{\oplus V} \oplus M^{\oplus E},$$

where M is the algebra of $n \times n$ matrices over k . The space $P_Q(n)$ is naturally an algebraic affine super-variety, where paths in $P_Q(n)$ of length 0 and 1 are given parities 0 and 1 respectively.

Definition 153 *The Schur quiver super-bialgebra, or Schiver super-bialgebra associated to (Q, n) , is the Schur super-bialgebra, $\mathcal{S}_Q(n) = \mathcal{S}(P_Q(n))$, associated to $P_Q(n)$. Its graded dual, the ring of functions,*

$$\mathcal{A}_Q(n) \cong (\mathcal{A}(M^*))^{\otimes V} \otimes \left(\bigwedge (M^*) \right)^{\otimes E},$$

on $P_Q(n)$, is isomorphic as an algebra, to a tensor product of symmetric and exterior algebras.

Our sole motivation for this definition is the apparent emergence of such structure in type A representation theory. But the alternating structure also bears encouraging homological consequences. For example, the supersymmetric aspect of this definition looks rather becoming, when one considers Koszul duality for these super-algebras - see remark 183.

Remark 154 When Q has merely one vertex, and no arrows, we recover the classical Schur bialgebras $\mathcal{S}(n)$ according to this construction. In general the tensor product,

$$\mathcal{S}_V(n) = (\mathcal{S}(n))^{\otimes V},$$

of Schur algebras, is naturally a sub-bialgebra of

$$\mathcal{S}_Q(n) \cong (\mathcal{S}(n))^{\otimes V} \otimes \left(\bigvee(n) \right)^{\otimes E}.$$

The inclusion map splits as an algebra homomorphism, via

$$\mathcal{S}_Q(n) \rightarrow \mathcal{S}_Q(n)/\mathcal{J} \cong (\mathcal{S}(n))^{\otimes V},$$

where \mathcal{J} is the direct sum of subspaces,

$$\left(\bigotimes_{v \in V} \mathcal{S}(n, a_v) \right) \otimes \left(\bigotimes_{e \in E} \bigvee(n, b_e) \right)$$

of $\mathcal{S}_Q(n)$, such that $b_e > 0$ for some $e \in E$. Thus, \mathcal{J} is a nilpotent ideal of the algebra $\mathcal{S}_Q(n)$.

If Q is a quiver, then the disjoint union $Q \amalg Q^{op}$ of quivers possesses an obvious orientation-reversing automorphism, exchanging Q and Q^{op} . Thus, $\mathcal{S}_{Q \amalg Q^{op}}(n)$ may be equipped with a k -endomorphism σ , which is an algebra and coalgebra anti-automorphism. Under such circumstances, we may apply theorem 142, and form the double, $D(\mathcal{S}_{Q \amalg Q^{op}}(n))$.

Definition 155 The Schiver double associated to (Q, n) , is the natural algebra summand,

$$\mathcal{D}_Q(n) = \mathcal{S}_Q(n) \otimes \mathcal{A}_{Q^{op}}(n),$$

of the double $D(\mathcal{S}_{Q \amalg Q^{op}}(n))$ corresponding to $Q \amalg Q^{op}$.

Thus, $\mathcal{D}_Q(n)$ is a direct sum, $\bigoplus_{r \geq 0} \mathcal{D}_Q(n, r)$ of algebras, where

$$\mathcal{D}_Q(n, r) = \bigoplus_{r_1 + r_2 = r} \mathcal{S}_Q(n, r_1) \otimes \mathcal{A}_{Q^{op}}(n, r_2).$$

Remark 156 Each algebra summand $\mathcal{D}_Q(n, r)$ is $\mathbb{Z}_+ \times \mathbb{Z}_+$ -graded, where the component,

$$\left(\bigotimes_{v \in V} \mathcal{S}(n, a_v) \right) \otimes \left(\bigotimes_{e \in E} V(n, b_e) \right) \\ \otimes \left(\bigotimes_{v \in V} \mathcal{A}(n, c_v) \right) \otimes \left(\bigotimes_{e \in E} \bigwedge(n, d_e) \right),$$

is given degree $(\sum_E b_e + \sum_V c_v, \sum_V c_v + \sum_E d_e)$.

Each algebra summand $\mathcal{D}_Q(n, r)$ is \mathbb{Z}_+ -graded, where the component above is given degree $(\sum_E b_e + 2\sum_V c_v + \sum_E d_e)$. We write $\mathcal{D}_Q^i(n, r)$ for the degree i part with respect to this grading. Thus,

$$\mathcal{D}_Q(n, r) = \bigoplus_{i=0}^{2r} \mathcal{D}_Q^i(n, r),$$

as a direct sum of graded pieces. In degree zero, we have

$$\mathcal{D}_Q^0(n, r) = \mathcal{S}_{V(Q)}(n, r).$$

Schiver doubles: independence of quiver orientation.

This section is devoted to a proof of the following result...

Theorem 157 *The Schiver double $\mathcal{D}_Q(n)$ is independent of the orientation of Q , and as such, is an invariant of the underlying graph of Q .*

For a locally finite graph Γ , we thus write $\mathcal{D}_\Gamma(n)$ for the Schiver double $\mathcal{D}_Q(n)$, where Q is any orientation of Γ .

We give a proof of theorem 157, in case $Q = A_1$ is the quiver with two vertices, and one arrow connecting those two vertices. To say that the corresponding double is independent of orientation, is to say that there is an algebra isomorphism between the double corresponding to the quiver, $\circ \longrightarrow \triangleright \circ$, and the double corresponding to the quiver, $\circ \triangleleft \longrightarrow \circ$. We therefore reveal an algebra automorphism of $\mathcal{D}_{A_1}(n)$.

Theorem 157 follows for a general quiver from the case $Q = A_1$. To see this, first observe that distinct arrows do not interact with one another when multiplied in $\mathcal{D}_Q(n)$. Therefore, if Q' is obtained from Q , by the reversing of an arrow, we have $\mathcal{D}_{Q'}(n) \cong \mathcal{D}_Q(n)$. Secondly note that we may obtain

one orientation of Q from another by reversing a collection of arrows, and so $\mathcal{D}_Q(n)$ is indeed independent of the orientation of Q .

We present a triad of preliminary lemmas.

Let $\vee(n) = \vee(M)$, and let $\wedge(n) = \wedge(M^*)$. Theorem 17 implies the following lemma.

Lemma 158 *There is an $\mathcal{S}(n)$ - $\mathcal{S}(n)$ -bimodule isomorphism, which exchanges $\vee(n)$ and $\wedge(n)$, for $n \geq 0$.*

When $n = 1$, this isomorphism is defined by the structure of a symmetric algebra on M . \square

We write $*$ for either of the inverse homomorphisms which describe the isomorphism of lemma 158. We have $(x \wedge y)^* = (x^* \vee y^*)$, and $(x \vee y)^* = (x^* \wedge y^*)$.

The tranpose anti-automorphism σ of $\mathcal{S}_{A_1}(n)$ maps $s \otimes \lambda \otimes s'$ to $s'^T \otimes \lambda^T \otimes s^T$.

Lemma 159 *The left action of $\mathcal{S}_{A_1}(n)$ on $\mathcal{A}_{A_1}(n)$, is given by,*

$$(s \otimes \lambda \otimes t) \circ (a \otimes \mu \otimes b) = \sum (-1)^{|\mu_{(2)}||\lambda|} (t_{(2)} \circ a) \otimes (t_{(1)} \circ \mu_{(2)}) \otimes (\mu_{(1)(2)} \cdot (s \circ b)) < \mu_{(1)(1)}, \lambda^T > .$$

The right action of $\mathcal{S}_{A_1}(n)$ on $\mathcal{A}_{A_1}(n)$, is given by,

$$(a \otimes \mu \otimes b) \circ (s \otimes \lambda \otimes t) = \sum (-1)^{|\mu_{(1)}||\mu_{(2)}|} ((a \circ t) \cdot \mu_{(2)(1)}) \otimes (\mu_{(1)} \circ s_{(2)}) \otimes (b \circ s_{(1)}) < \mu_{(2)(2)}, \lambda^T > .$$

Proof:

We record a calculation for the left action:

$$\begin{aligned} & < (s \otimes \lambda \otimes t) \circ (a \otimes \mu \otimes b), (s' \otimes \lambda' \otimes t') > \\ &= < a \otimes \mu \otimes b, (s \otimes \lambda \otimes t)^\sigma \circ (s' \otimes \lambda' \otimes t') > \\ &= < a \otimes \mu \otimes b, (t^T \otimes \lambda^T \otimes s^T) \circ (s' \otimes \lambda' \otimes t') > \\ &= \sum < a \otimes \mu \otimes b, (t^T)_{(1)} s' \otimes (\lambda^T \circ t'_{(1)}) \vee ((t^T)_{(2)} \circ \lambda') \otimes s^T t'_{(2)} > \\ &= \sum (-1)^{|\mu_{(2)}||\lambda|} < a, t_{(2)}^T s' > < \mu_{(1)}, (\lambda^T \circ t'_{(1)}) > \\ & \quad < \mu_{(2)}, (t_{(1)}^T \circ \lambda') > < b, s^T t'_{(2)} > \\ &= \sum (-1)^{|\mu_{(2)}||\lambda|} < t_{(2)} \circ a, s' > < \mu_{(1)(1)}, \lambda^T > < \mu_{(1)(2)}, t'_{(1)} > \\ & \quad < t_{(1)} \circ \mu_{(2)}, \lambda' > < s \circ b, t'_{(2)} > \\ &= \sum (-1)^{|\mu_{(2)}||\lambda|} < (t_{(2)} \circ a \otimes t_{(1)} \circ \mu_{(2)} \otimes \mu_{(1)(2)} \cdot (s \circ b)), (s' \otimes \lambda' \otimes t') > \end{aligned}$$

$$< \mu_{(1)(1)}, \lambda^T > \square$$

Dual to the bilinear form

$$\epsilon : \bigwedge(n) \otimes \bigvee(n) \rightarrow k,$$

there is a natural map

$$\epsilon^* : k \rightarrow \bigwedge(n) \otimes \bigvee(n).$$

Squeezing the identity map on $\mathcal{S}(n)$ inside ϵ , we obtain a map

$$\phi : \mathcal{S}(n) \rightarrow \bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n).$$

The right action of $\mathcal{S}(n)$ on $\bigwedge(n)$ may be formulated as a map,

$$m_1 : \bigwedge(n) \otimes \mathcal{S}(n) \rightarrow \bigwedge(n).$$

The left action of $\mathcal{S}(n)$ on $\bigvee(n)$ may be formulated as a map,

$$m_2 : \mathcal{S}(n) \otimes \bigvee(n) \rightarrow \bigvee(n).$$

Lemma 160 *The diagram*

$$\begin{array}{ccc} \mathcal{S}(n) & \xrightarrow{\phi} & \bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n) \\ \downarrow \phi & & \downarrow T \otimes m_2 \\ \bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n) & \xrightarrow{m_1 \otimes T} & \bigwedge(n) \otimes \bigvee(n) \end{array}$$

commutes.

Proof:

Let $\{\xi_{ij} \mid i, j = 1, \dots, n\}$ be a basis for M . Let $\{X_{ij}\}$ be the basis for M^* which is identified with $\{\xi_{ij}\}$ via the symmetric structure on M . The Schur algebra may be given basis, whose elements have the form $\xi_{a_1 b_1} \dots \xi_{a_r b_r}$. Let $\Sigma(a, b)$ be the stabilizer in Σ_n of the sequence $(a_i, b_i)_{i=1}^n$. We have,

$$\begin{aligned} & (m_1 \otimes T) \circ \phi(\xi_{a_1 b_1} \dots \xi_{a_r b_r}) \\ &= (m_1 \otimes 1) \left(\sum_{i_k, j_k} X_{i_1 j_1} \wedge \dots \wedge X_{i_n j_n} \otimes \xi_{a_1 b_1} \dots \xi_{a_r b_r} \otimes \xi_{j_1 i_1} \vee \dots \vee \xi_{j_r i_r} \right) \\ &= (m_1 \otimes 1) \left(\sum_{i_k, j_k, \sigma \in \Sigma(a, b)} X_{i_1 j_1} \wedge \dots \wedge X_{i_n j_n} \otimes \xi_{a_{\sigma 1} b_{\sigma 1}} \otimes \dots \otimes \xi_{a_{\sigma r} b_{\sigma r}} \otimes \right. \\ & \quad \left. \xi_{j_1 i_1} \vee \dots \vee \xi_{j_r i_r} \right) \end{aligned}$$

$$= \sum_{i_k, \sigma \in \Sigma(a, b)} X_{i_1 b_{\sigma_1}} \wedge \dots \wedge X_{i_n b_{\sigma_n}} \otimes \xi_{a_{\sigma_1 i_1}} \vee \dots \vee \xi_{a_{\sigma_r i_r}}.$$

A similar computation shows that $(T \otimes m_2) \circ \phi$ is equal to the same sum. \square

The duals of m_1, m_2 are maps,

$$m_1^* : \bigvee(n) \rightarrow \bigvee(n) \otimes \mathcal{A}(n).$$

$$m_2^* : \bigwedge(n) \rightarrow \mathcal{A}(n) \otimes \bigwedge(n).$$

We write $m_1^*(\gamma) = \sum \gamma_{(1)} \otimes \gamma_{(2)}$ for $\gamma \in \bigvee(n)$, and $m_2^*(\delta) = \sum \delta_{(1)} \otimes \delta_{(2)}$ for $\delta \in \bigwedge(n)$.

Lemma 161 *For $\alpha \in \bigvee(n), \beta \in \bigwedge(n)$, we have,*

$$\sum \alpha_{(2)} < \alpha_{(1)}, \beta^T > = \sum < \alpha^{*T}, \beta_{(2)}^*, > \beta_{(1)}^*$$

Proof:

By lemma 160, the following diagram commutes:

$$\begin{array}{ccccc}
& & \mathcal{S}(n) & & \\
& \swarrow \phi & \downarrow \phi & \searrow (* \otimes 1 \otimes *) \phi & \\
\bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n) & & \bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n) & & \bigvee(n) \otimes \mathcal{S}(n) \otimes \bigwedge(n) \\
\downarrow m_1 \otimes T & & \swarrow * \otimes 1 \otimes * & & \downarrow T \otimes m_2 \\
\bigwedge(n) \otimes \bigvee(n) & \xrightarrow{T \otimes m_2} & \bigwedge(n) \otimes \mathcal{S}(n) \otimes \bigvee(n) & \xrightarrow{* \otimes 1 \otimes *} & \bigvee(n) \otimes \mathcal{S}(n) \otimes \bigwedge(n) \\
& \xrightarrow{* \otimes *} & & & \\
& \bigwedge(n) \otimes \bigvee(n) & \xrightarrow{* \otimes *} & \bigvee(n) \otimes \bigwedge(n) &
\end{array}$$

Therefore, the diagram dual to this one commutes. The two passages from $\bigvee(n) \otimes \bigwedge(n)$ to $\mathcal{A}(n)$ around the boundary of this dual diagram describe the two sides to the formula of the lemma. \square

Theorem 162 *There is an involutory algebra automorphism θ of $\mathcal{D}_{A_1}(n)$, given by,*

$$\theta(s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b) = (-1)^{|\lambda||\xi|} (t \otimes \xi^* \otimes s \otimes b \otimes \lambda^* \otimes a).$$

Proof:

$$\begin{aligned}
& [s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b] \cdot [u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d] \\
& = \sum \pm (s \otimes \lambda \otimes t)_{(2)} \cdot (u \otimes \mu \otimes v)_{(1)} \otimes [(s \otimes \lambda \otimes t)_{(1)} \circ (c \otimes \eta \otimes d)] \otimes
\end{aligned}$$

$$\begin{aligned}
& [(a \otimes \xi \otimes b) \circ (u \otimes \mu \otimes v)_{(2)}] \\
= & \sum \pm (s_{(2)} \otimes \lambda_{(2)} \otimes t_{(2)}) \cdot (u_{(1)} \otimes \mu_{(1)} \otimes v_{(1)}) \otimes \\
& [(s_{(1)} \otimes \lambda_{(1)} \otimes t_{(1)}) \circ (c \otimes \eta \otimes d)] \otimes \\
& [(a \otimes \xi \otimes b) \circ (u_{(2)} \otimes \mu_{(2)} \otimes v_{(2)})] \\
= & \sum \pm s_{(2)(1)} u_{(1)} \otimes (\lambda_{(2)} \circ v_{(1)(1)}) \vee (s_{(2)(2)} \circ \mu_{(1)}) \otimes t_{(2)} v_{(1)(2)} \otimes \\
& [t_{(1)(2)} \circ c \otimes t_{(1)(1)} \circ \eta_{(2)} \otimes \eta_{(1)(2)} \cdot (s_{(1)} \circ d)] \cdot \\
& [(a \circ v_{(2)}) \cdot \xi_{(2)(1)} \otimes \xi_{(1)} \circ u_{(2)(2)} \otimes c \circ u_{(2)(1)}] \\
& < \eta_{(1)(1)}, \lambda_{(1)}^T > < \xi_{(2)(2)}, \mu_{(2)}^T > \\
= & \sum \pm s_{(2)} u_{(1)} \otimes (\lambda_{(2)} \circ v_{(1)}) \vee (s_{(3)} \circ \mu_{(1)}) \otimes t_{(3)} v_{(2)} \otimes \\
& [t_{(2)} \circ c \otimes t_{(1)} \circ \eta_{(3)} \otimes \eta_{(2)} \cdot (s_{(1)} \circ d)] \cdot \\
& [(a \circ v_{(3)}) \cdot \xi_{(2)} \otimes \xi_{(1)} \circ u_{(3)} \otimes c \circ u_{(2)}] \\
& < \eta_{(1)}, \lambda_{(1)}^T > < \xi_{(3)}, \mu_{(2)}^T > \\
= & \sum \pm s_{(2)} u_{(1)} \otimes (\lambda_{(2)} \circ v_{(1)}) \vee (s_{(3)} \circ \mu_{(1)}) \otimes t_{(3)} v_{(2)} \otimes \\
& (t_{(2)} \circ c) \cdot (a \circ v_{(3)}) \cdot \xi_{(2)} \otimes (t_{(1)} \circ \eta_{(3)}) \wedge (\xi_{(1)} \circ u_{(3)}) \otimes \\
& \eta_{(2)} \cdot (s_{(1)} \circ d) \cdot (c \circ u_{(2)}) < \eta_{(1)}, \lambda_{(1)}^T > < \xi_{(3)}, \mu_{(2)}^T > .
\end{aligned}$$

Thus,

$$\begin{aligned}
& \theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]) \cdot \theta([u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d]) \\
= & [t \otimes \xi^* \otimes s \otimes b \otimes \lambda^* \otimes a] \cdot [v \otimes \eta^* \otimes u \otimes d \otimes \mu^* \otimes c] \\
= & \sum \pm t_{(2)} v_{(1)} \otimes (\xi_{(2)}^* \circ u_{(1)}) \vee (t_{(3)} \circ \eta_{(1)}^*) \otimes s_{(3)} u_{(2)} \otimes \\
& (s_{(2)} \circ d) \cdot (b \circ u_{(3)}) \cdot \lambda_{(2)}^* \otimes (s_{(1)} \circ \mu_{(3)}^*) \wedge (\lambda_{(1)}^* \circ v_{(3)}) \otimes \\
& \mu_{(2)}^* \cdot (t_{(1)} \circ c) \cdot (d \circ v_{(2)}) < \mu_{(1)}^*, \xi_{(1)}^{*T} > < \lambda_{(3)}^*, \eta_{(2)}^{*T} > .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]) \cdot [u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d] \\
= & \sum \pm t_{(3)} v_{(2)} \otimes [(t_{(1)} \circ \eta_{(3)}) \wedge (\xi_{(1)} \circ u_{(3)})]^* \otimes s_{(2)} u_{(1)} \otimes \\
& \eta_{(2)} \cdot (s_{(1)} \circ d) \cdot (c \circ u_{(2)}) \otimes [(\lambda_{(2)} \circ v_{(1)}) \vee (s_{(3)} \circ \mu_{(1)})]^* \otimes \\
& (t_{(2)} \circ c) \cdot (a \circ v_{(3)}) \cdot \xi_{(2)} < \eta_{(1)}, \lambda_{(1)}^T > < \xi_{(3)}, \mu_{(2)}^T > . \\
= & \sum \pm t_{(3)} v_{(2)} \otimes (\xi_{(1)}^* \circ u_{(3)}) \vee (t_{(1)} \circ \eta_{(3)}^*) \otimes s_{(2)} u_{(1)} \otimes \\
& \eta_{(2)} \cdot (s_{(1)} \circ d) \cdot (c \circ u_{(2)}) \otimes (s_{(3)} \circ \mu_{(1)}^*) \wedge (\lambda_{(2)}^* \circ v_{(1)}) \otimes \\
& (t_{(2)} \circ c) \cdot (a \circ v_{(3)}) \cdot \xi_{(2)} < \eta_{(1)}, \lambda_{(1)}^T > < \xi_{(3)}, \mu_{(2)}^T > .
\end{aligned}$$

Given the cocommutativity of the classical Schur algebra, it is clear from lemma 161, that up to a sign, the terms in our expression for

$$\theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]) \cdot [u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d]$$

agree with the terms in our expression for

$$\theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]) \cdot \theta([u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d]).$$

The difference in sign of each term is precisely

$$(-1)^{|\lambda_{(1)}||\lambda_{(2)}|+|\mu_{(1)}||\mu_{(2)}||\xi_{(1)}||\xi_{(2)}|+|\eta_{(1)}||\eta_{(2)}|}.$$

Diagrammatically comparing a term of

$$\theta(\theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]).\theta([u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d]))$$

with the relevant term of

$$[s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b].[u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d],$$

one sees that their difference is also

$$(-1)^{|\lambda_{(1)}||\lambda_{(2)}|+|\mu_{(1)}||\mu_{(2)}||\xi_{(1)}||\xi_{(2)}|+|\eta_{(1)}||\eta_{(2)}|}.$$

Therefore, our expressions for

$$\theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b].[u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d]),$$

and

$$\theta([s \otimes \lambda \otimes t \otimes a \otimes \xi \otimes b]).\theta([u \otimes \mu \otimes v \otimes c \otimes \eta \otimes d])$$

agree. This completes the proof of theorem 162. \square

Schiver doubles and wreath products.

Schiver bialgebras and their doubles may be understood to be generalisations of certain wreath products, as is illustrated by the following example:

Let $\mathcal{S}(n)$ be the Schur bialgebra associated to $GL_n(k)$, and let the Schur algebra $\mathcal{S}(n, r)$ be the subalgebra of degree r . Let the double of $\mathcal{S}(n)$ be denoted $\mathcal{D}(n)$, and let its degree r part be written $\mathcal{D}(n, r)$.

Let $n \geq r$, and let $\omega = (1^r)$ be the partition of r with r parts. According to Green's presentation of Schur-Weyl duality (theorem 23),

$$\xi_\omega \mathcal{S}(n, r) \xi_\omega \cong k\Sigma_r.$$

We have the following generalisation (for a further generalisation, see [69], theorem 3):

Proposition 163 (a) *The endomorphism ring $\xi_\omega \mathcal{D}(n, r) \xi_\omega$ is isomorphic to the wreath product $k[x]/(x^2) \wr \Sigma_r$.*

(b) *If $\text{char}(k) = 2$, then $\xi_\omega \mathcal{D}(n, r) \xi_\omega$ is isomorphic to the wreath product $k\Sigma_2 \wr \Sigma_r$.*

(c) *If $\text{char}(k) = 0$, or $w < \text{char}(k)$, then $\mathcal{D}(n, r)$ is Morita equivalent to $k[x]/(x^2) \wr \Sigma_r$.*

Proof:

Let $\Omega = \{1, \dots, r\}$. Let us identify a subset $\Pi \subset \Omega$ of size d with the d -tuple $(\pi_1, \dots, \pi_d) \in I(n, d)$, where Π is ordered just as Ω is ordered.

Thus, $\xi_{\Omega, \Omega} = \xi_\omega$. The set

$$\{\xi_{\Pi\sigma, \Pi} \otimes X_{(\Omega-\Pi)\sigma, (\Omega-\Pi)} \mid \Pi \subset \Omega, \sigma \in \Sigma_r\}$$

is a basis for $\xi_\omega D(n, r)\xi_\omega$. The size of this basis set is $2^r \cdot r!$.

The subspace spanned by $\{\xi_{\Omega\sigma, \Omega} \mid \sigma \in \Sigma_r\}$ is a subalgebra, naturally isomorphic to $k\Sigma_r$.

For $\Pi \subset \Omega$, let $\xi_\Pi = \xi_{\Pi, \Pi}$. We show that the subspace spanned by $\{\xi_\Pi \otimes X_{\Omega-\Pi} \mid \Pi \subset \Omega\}$ is also a subalgebra, isomorphic to $(k[x]/(x^2))^{\otimes r}$ via

$$\Phi : \xi_\Pi \otimes X_{\Omega-\Pi} \mapsto z_1 \otimes \dots \otimes z_r,$$

where $z_i = 1$ if $i \in \Pi$, and $z_i = 0$ otherwise.

The isomorphism of the lemma is then quite plain from the formulas,

$$\xi_\Pi \otimes X_{\Omega-\Pi} \cdot \xi_{\Omega\sigma, \Omega} = \xi_{\Pi\sigma, \Pi} \otimes X_{\Omega-\Pi\sigma, \Omega-\Pi},$$

$$\xi_{\Omega\sigma^{-1}, \Omega} \cdot \xi_\Pi \otimes X_{\Omega-\Pi} \cdot \xi_{\Omega\sigma, \Omega} = \xi_\Pi \otimes X_{(\Omega-\Pi)\sigma}.$$

Let Γ, Π be subsets of Ω . Let $s = \xi_\Pi$, and let $a = X_{\Omega-\Pi}$. Let $t = \xi_\Gamma$, and let $b = X_{\Omega-\Gamma}$. We compute the product of $s \otimes a$ and $t \otimes b$.

$$\Delta(\xi_\Pi) = \sum_{\alpha \subset \Pi} \xi_\alpha \otimes \xi_{\Pi-\alpha}$$

$$\Delta(\xi_\Gamma) = \sum_{\beta \subset \Gamma} \xi_\beta \otimes \xi_{\Gamma-\beta}.$$

Thus, non-zero terms of $(s \otimes a)(t \otimes b)$ only appear when $b_{(1)} = X_\alpha$, $s_{(2)} = \xi_{\Pi-\alpha}$, $a_{(2)} = X_{\Gamma-\beta}$, and $t_{(1)} = \xi_\beta$.

This implies, in turn, that only $b = b_{(1)} = b_{(2)} = X_\alpha$ gives possible non-zero terms in $(s \otimes a)(t \otimes b)$. Thus, $\alpha = \Omega - \Gamma$.

In addition, only $a = a_{(1)} = a_{(2)} = X_{\Gamma-\beta}$ gives possible non-zero terms in $(s \otimes a)(t \otimes b)$. Thus, $\Gamma - \beta = \Omega - \Pi$.

For the product $s_{(2)}t_{(1)}$ to be non-zero, we now require $\beta = \Pi - \alpha$. Thus, $\beta = \Gamma \cap \Pi$ and $\alpha = \Omega - \Gamma$.

From all this, we conclude that $(s \otimes a)(t \otimes b)$ is zero unless $\Pi \cup \Gamma = \Omega$, and that in this case $(s \otimes a)(t \otimes b)$ is equal to $\xi_{\Pi \cap \Gamma} \otimes X_{\Omega - \Pi \cap \Gamma}$. It follows that Φ is indeed an algebra isomorphism, and (a) is proven !

(b) is immediate from (a), since $k\Sigma_2 \cong k[x]/(x^2)$, in characteristic two.

(c) also follows immediately from (a), because $k\Sigma_r$ is semisimple so long as $r!$ is invertible in k . Thus, $\mathcal{D}(n, r)$ and $k[x]/(x^2) \wr \Sigma_r$ have the same number of simple modules. \square

Schiver doubles and blocks of Hecke algebras.

Here is a summary of our education concerning Rock blocks of symmetric groups, in characteristic two, in chapters 4 and 5.

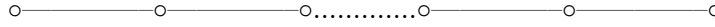
Theorem 164 *Let k be a field of characteristic two. Let $k\mathbf{B}_{\rho,w}^\Sigma$ be a Rock block of a symmetric group, whose weight is w . Then $k\mathbf{B}_{\rho,w}^\Sigma$ contains a nilpotent ideal \mathcal{N} , such that $k\mathbf{B}_{\rho,w}^\Sigma/\mathcal{N}$ is Morita equivalent to $\mathcal{S}(w, w)$. Let e be an idempotent in $k\mathbf{B}_{\rho,w}^\Sigma$, such that the indecomposable summands of $k\mathbf{B}_{\rho,w}^\Sigma e$ are precisely those indecomposable summands with tops in the set,*

$$\{D^{[\emptyset, \lambda]} | \lambda \text{ is } p\text{-regular} \}.$$

Then $ek\mathbf{B}_{\rho,w}^\Sigma e$ is Morita equivalent to $k\Sigma_2 \wr \Sigma_w$. \square

This theorem, proposition 163, and indeed example 147 of this paper, give evidence that Rock blocks $k\mathbf{B}_{\rho,w}^\Sigma$, are in fact Morita equivalent to the Schur doubles $\mathcal{D}(n, w)$, for $n \geq w$, when the field k has characteristic two. But the Ringel dualities of chapter 6 impose a more general conjecture on us:

Let A_{p-1} be the ordinary Dynkin graph with $p - 1$ vertices:



Let $q \in k^\times$, and let p be the least natural number such that,

$$1 + q + \dots + q^{p-1} = 0.$$

Let n, w be natural numbers, such that $n \geq w$. Let $\mathcal{D}_{A_{p-1}}(n)$ be the Schiver double associated to the graph A_{p-1} .

Conjecture 165 *The degree w part $\mathcal{D}_{A_{p-1}}(n, w)$ of $\mathcal{D}_{A_{p-1}}(n)$, is Morita equivalent to the Rock block $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ of a Hecke algebra, whose weight is w .*

Indeed, $\mathcal{D}_{A_{p-1}}(n, w)$ is derived equivalent to any block $k\mathbf{B}_{\tau,w}^{\mathcal{H}_q}$ of a Hecke algebra, whose weight is w .

Since any two blocks of the same weight are derived equivalent, the conjectured derived equivalences would follow immediately from the conjectured Morita equivalences.

Remark 166 When $q = 1$, the Hecke algebra $\mathcal{H}_q(\Sigma_n)$ is isomorphic to the symmetric group algebra $k\Sigma_n$, and p is merely the characteristic of the field

k . In this case, conjecture 165 could be viewed as relating to a non-abelian generalisation of Broué's abelian defect group conjecture [9] for symmetric groups, as proved by Chuang, Kessar, and Rouquier [11], [12].

So far, however, I can see no natural interpretation of the Schiver double $\mathcal{D}_{A_{p-1}}(n)$ in terms of p -local group theoretic information, weights for symmetric groups, etc. etc.

Doubles, and graded rings associated to Rock blocks of Hecke algebras.

I am unable to prove conjecture 165. The best I can do, is describe a filtration,

$$k\mathbf{B}_{\rho,w}^{\mathcal{H}_q} = \mathcal{N}[0] \supset \mathcal{N}[1] \supset \mathcal{N}[2] \supset \dots \supset \mathcal{N}[2w] \supset \mathcal{N}[2w+1] = 0,$$

on the Rock block of a Hecke algebra of weight w , such that $\mathcal{N}[i]\mathcal{N}[j] \subseteq \mathcal{N}[i+j]$, and the associated graded ring,

$$gr_{\rho,w}^{\mathcal{H}_q} = \bigoplus_{i=0}^{2w} \mathcal{N}[i]/\mathcal{N}[i+1],$$

resembles the Schiver double $\mathcal{D}_{A_{p-1}}(w, w)$. Suppose $p \geq 3$. We can show that the degree zero part of the graded ring $gr_{\rho,w}^{\mathcal{H}_q}$ is Morita equivalent to the degree zero part $\mathcal{S}_{V(A_{p-1})}(w, w)$ of $\mathcal{D}_{A_{p-1}}(w, w)$, and that the homogeneous components of $gr_{\rho,w}^{\mathcal{H}_q}$ correspond via Morita equivalence, in the category of $\mathcal{S}_{V(A_{p-1})}(w, w)$ - $\mathcal{S}_{V(A_{p-1})}(w, w)$ -bimodules, to the homogeneous components of $\mathcal{D}_{A_{p-1}}(w, w)$.

In this section, we give a sketch of a proof of this fact. The most obvious obstacle to proving conjecture 165, concerning Rock blocks of Hecke algebras, is our inability to show that $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ is graded, in a certain way.

Step 1. If K is a certain field of characteristic zero, and $q \in K$ is a primitive p^{th} root of unity, then $K\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$ is Morita equivalent to $K\mathbf{B}_{\emptyset,1}^{\mathcal{H}_q} \wr \Sigma_w$.

Proof:

By Dirichlet's theorem, there exists a prime number l , such that $l \equiv 1$ (modulo p). Similarly, there exists a prime number \bar{q} , such that $\bar{q} \equiv q$ (modulo l). By theorem 80, there is an l -modular system (K, \mathcal{O}, k) , such that $k\mathbf{B}_{\rho,w}^{\mathcal{H}_{\bar{q}}}$ is Morita equivalent to $k\mathbf{B}_{\emptyset,1}^{\mathcal{H}_{\bar{q}}} \wr \Sigma_w$. We can lift this equivalence by the following argument, due to Joe Chuang.

The bimodule inducing Morita equivalence is a summand of the $k\mathbf{B}_{\rho,w}^{\mathcal{H}_{\bar{q}}}$ - $k\mathbf{B}_{\emptyset,1}^{\mathcal{H}_{\bar{q}}} \wr \Sigma_w$ -bimodule,

$$kT = k\mathbf{B}_{\rho,w}^{\mathcal{H}_{\bar{q}}} \bigotimes_{\otimes^w k\mathbf{B}_{\emptyset,1}^{\mathcal{H}_{\bar{q}}}} k\mathbf{B}_{\emptyset,1}^{\mathcal{H}_{\bar{q}}} \wr \Sigma_w.$$

We would like to define a summand of the $\mathcal{OB}_{\rho,w}^{\mathcal{H}_q}$ - $\mathcal{OB}_{\emptyset,1}^{\mathcal{H}_q} \wr \Sigma_w$ -bimodule,

$$T = \mathcal{OB}_{\rho,w}^{\mathcal{H}_q} \bigotimes_{\otimes^w K\mathbf{B}_{\emptyset,1}^{\mathcal{H}_q}} \mathcal{OB}_{\emptyset,1}^{\mathcal{H}_q} \wr \Sigma_w,$$

which induces a Morita equivalence. It is enough lift idempotents from $\text{End}(kT)$ to $\text{End}(T)$. Algebraically, this translates to the problem of lifting centralizers of parabolic subalgebras from characteristic l to characteristic zero. The arguments of A. Francis ([32], 3.6, 3.8) show that this is possible.

Step 2. Use the approach of Cline, Parshall, and Scott ([19], 5.3) to generalise theorem 90 from blocks of symmetric groups to blocks of Hecke algebras. Generalise the results of chapters 6 and 7 from blocks of symmetric groups, to blocks of Hecke algebras. Thus produce a nilpotent ideal \mathcal{N} of $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$, such that $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ is Morita equivalent to $\mathcal{S}_{V(A_{p-1})}(w, w)$.

Step 3. Define the filtration $\mathcal{N}[i]$ on $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}$, using good idempotents for the q -Schur algebra, as well as the signature automorphism on the Hecke algebra.

The method of definition generalises that of the bimodule N , of chapter 6. Whilst N is defined to be $xk\mathbf{B}_{\rho,w}^{\mathcal{H}_q}y$, for fixed idempotents x, y , the ideal $\mathcal{N}[i]$ contains sums of terms $x\mathcal{H}_qy$, $y\mathcal{H}_qx$, $x\mathcal{H}_qy\mathcal{H}_qx$, and $y\mathcal{H}_qx\mathcal{H}_qy$, for various idempotents x, y .

By comparison with the characteristic zero case (Step 1), it can be seen that the filtration satisfies $\mathcal{N}[i]\mathcal{N}[j] \subseteq \mathcal{N}[i+j]$, and $\mathcal{N} = \mathcal{N}[1]$.

Step 4. Show that the $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ - $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ -bimodule, $\mathcal{N}[i]/\mathcal{N}[i+1]$ corresponds, via Morita equivalence, to the $\mathcal{D}_{A_{p-1}}^0(w, w)$ - $\mathcal{D}_{A_{p-1}}^0(w, w)$ -bimodule, $\mathcal{D}_{A_{p-1}}^i(w, w)$. Prove this by induction on w .

Steps 1-4 imply that the graded components of the algebras $gr_{\rho,w}^{\mathcal{H}_q}$ and $\mathcal{D}_{A_{p-1}}(w, w)$ are in natural correspondence. A more ambitious project would be to follow through steps 5-8, and thus prove that $gr_{\rho,w}^{\mathcal{H}_q}$ and $\mathcal{D}_{A_{p-1}}(w, w)$ are Morita equivalent. There are possible difficulties in pushing this through.

In an analogous, but more elementary situation, we have succeeded in overcoming the necessary obstacles, in work with V. Miemietz [56].

Step 5. Show that for compositions $(a_1, \dots, a_{p-1}), (c_1, \dots, c_{p-1})$ of w , for which there exists a composition (b_1, \dots, b_{p-1}) of w , such that,

$$(a_1, \dots, a_{p-1}) \triangleleft (b_1, \dots, b_{p-1}) \triangleleft (c_1, \dots, c_{p-1}),$$

we have,

$$\begin{aligned} \text{Ext}^1(\mathcal{L}_q(a_1, \dots, a_{p-1}), \mathcal{L}_q(c_1, \dots, c_{p-1})) = \\ \text{Ext}^1(\mathcal{L}_q(c_1, \dots, c_{p-1}), \mathcal{L}_q(a_1, \dots, a_{p-1})) = 0. \end{aligned}$$

Prove this by induction on w , using duality and Frobenius reciprocity, with base case $w = 1$.

Step 6. Show (using Step 5) that the multiplication morphism,

$$\phi : \mathcal{N}[i]/\mathcal{N}[i+1] \bigotimes_{k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}} \mathcal{N}[j]/\mathcal{N}[j+1] \rightarrow \mathcal{N}[i+j]/\mathcal{N}[i+j+1],$$

of $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ - $k\mathbf{B}_{\rho,w}^{\mathcal{H}_q}/\mathcal{N}$ -bimodules, is a surjection. In other words, $gr_{\rho,w}^{\mathcal{H}_q}$ is generated in degrees 0 and 1.

Step 7 The generalized Koszulity of $\mathcal{D}_{A_\infty}(w, w)$ (remark 183) implies that $\mathcal{D}_{A_\infty}(w, w)$ is quadratic. When $p \geq 4$, $\mathcal{D}_{A_{p-1}}(w, w)$ is also quadratic.

The correspondences of Step 4 may be fixed so that in degree $(1, 1)$, ϕ corresponds to the morphism

$$\begin{aligned} \mathcal{D}_{A_{p-1}}^1(w, w) \bigotimes_{\mathcal{D}_{A_{p-1}}^0(w, w)} \mathcal{D}_{A_{p-1}}^1(w, w) \\ \rightarrow \mathcal{D}_{A_{p-1}}^2(w, w). \end{aligned}$$

of $\mathcal{D}_{A_{p-1}}^0(w, w)$ - $\mathcal{D}_{A_{p-1}}^0(w, w)$ -bimodules.

Step 8. Conclude from Steps 6 and 7 that $\mathcal{D}_{A_{p-1}}(w, w)$ surjects onto an algebra Morita equivalent to $gr_{\rho,w}^{\mathcal{H}_q}$. By a dimension count, this surjection is an isomorphism. \square

Chapter IX

Power sums.

In this chapter, we define certain chain complexes for the Schiver doubles, whose Grothendieck character describes the symmetric functions $p_r = x_1^r + x_2^r + \dots$ (theorem 168).

Of course, it is not difficult to define such complexes in a naive way: take standard modules for the Schur algebra indexed by hook partitions, place them in homological degree i , where i is the number of parts of the partition, and then give them a zero differential.

However, we describe here a more subtle method, which uses the structure of the doubles, rather than merely the Schur algebra. The reason for expecting such complexes to exist, and defining them, is the categorification program. Indeed, the existence of such complexes, which invoke the structure of the Schiver doubles, is consistent with the apparent affinity between derived categories of Schiver doubles, and those of blocks of Hecke algebras.

Whilst blocks of Hecke algebras define a category lifting the Fock space realization of the basic representation of $\widehat{\mathfrak{sl}}_p$, power sums play a defining role in the combinatorial formulation of the principal homogeneous realization of the basic representation for $\widehat{\mathfrak{sl}}_p$ by I. Frenkel, N. Jing and W. Wang. We expect the equivalences between blocks of Hecke algebras and doubles to be one aspect of a categorical realisation of the isomorphism between the Fock space realization and the principal homogeneous realization. The description of induction and restriction functors between symmetric groups via certain functors between doubles should be another aspect. Indeed, we expect functors between doubles which correspond to power sums. Note that such functors should be realised between doubles, and not their Schur algebra quotients, because it is the doubles which we expect to categorify the principal homogeneous realization, and not their quotients.

Complexes for Schiver doubles, and power sums.

Most of the bases for the ring of symmetric functions given in I. Macdonald's book [54] have natural interpretations as characters of modules for the Schur algebra. Elementary symmetric functions correspond to exterior powers of the natural module for $M_n(k)$, complete symmetric functions correspond to symmetric powers of the natural module, and Schur functions correspond to Weyl modules. However, the power sums $p_r = x_1^r + x_2^r + \dots$ have no such interpretation.

In this section, we describe complexes \mathcal{P}_r for Schiver doubles $\mathcal{D}_\Gamma(n, r)$, whose homology describes the power sum p_r . Indeed, we may define one

such complex $\mathcal{P}_r(a)$, for every pair (γ, a) , where γ is an vertex of Γ , and a an edge emanating from γ .

Lemma 167 *Let $n \geq r$. Let Q be a quiver. Suppose that γ_1, γ_2 are vertices in Q . Suppose that a is an arrow in Q , whose source is γ_1 , and tail is γ_2 , and that a is the only such arrow. Let r_1, r_2 be natural numbers, whose sum is r . There is a graded module $M_{r_1, r_2} = M_{r_1, r_2}(a)$ for $\mathcal{S}_Q(n, r)$, whose graded pieces are the $\mathcal{S}_{V(Q)}(n, r)$ -modules,*

$$\begin{aligned} & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \Delta_{\gamma_1}(r_1, 1) \otimes E_{\gamma_2}^{\otimes r_2-1} \\ & \dots \\ & \Delta_{\gamma_1}(r_1, 1^{r_2}), \end{aligned}$$

in degrees $0, 1, \dots, r_2$.

Proof:

Let ξ_{r_1, r_2} be the unit of the algebra $\mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)$, an idempotent in $\mathcal{S}_Q(n, r)$. Then,

$$N_{r_1, r_2}(a) = \mathcal{S}_Q(n, r) \xi_{r_1, r_2} \bigotimes_{\mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)} \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2},$$

is a graded module for $\mathcal{S}_Q(n, r)$, whose graded pieces are,

$$\begin{aligned} & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_1} \otimes E_{\gamma_2}^{\otimes r_2-1} \\ & \dots \\ & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_1}^{\otimes r_2}, \end{aligned}$$

in degrees $0, 1, \dots, r_2$.

For a partition λ of t , let $\xi_{>\lambda} = \sum_{\mu > \lambda} \xi_\mu$ be the sum of Green's idempotents ξ_μ [38], corresponding to partitions μ , greater than λ , with respect to the dominance ordering.

Let i_{r_1, r_2}^j be the idempotent $\xi_{>(r_1, 1^j)} \otimes \xi_{(1^{r_2-j})}$, an element of the algebra,

$$\mathcal{S}_{\gamma_1}(n, r_1 + j) \otimes \mathcal{S}_{\gamma_2}(n, r_2 - j).$$

Let $i_{r_1, r_2} = \sum_{j=1}^{r_2} i_{r_1, r_2}^j$, an idempotent in the algebra,

$$\bigoplus_{j=1}^{r_2} \mathcal{S}_{\gamma_1}(n, r_1 + j) \otimes \mathcal{S}_{\gamma_2}(n, r_2 - j).$$

Let us define,

$$M_{r_1, r_2}(a) = N_{r_1, r_2}(a) / \mathcal{S}_Q(n, r) i_{r_1, r_2} N_{r_1, r_2}(a),$$

to be the quotient of $N_{r_1, r_2}(a)$, relative to a trace from the projective module $\mathcal{S}_Q(n, r) i_{r_1, r_2}$.

Since the tensor product of a projective $\mathcal{S}_{\gamma_1}(n, r_1 + j)$ -module with the natural representation E_{γ_1} is projective, with Δ -composition factors given by the branching rule, we may compute the composition factors of $M_{r_1, r_2}(a)$, and find them to be,

$$\begin{aligned} & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \Delta_{\gamma_1}(r_1, 1) \otimes E_{\gamma_2}^{\otimes r_2 - 1} \\ & \dots \\ & \Delta_{\gamma_1}(r_1, 1^{r_2}), \end{aligned}$$

in degrees $0, 1, \dots, r_2$. \square

The following theorem generalises the above lemma:

Theorem 168 *Let $n \geq r$. Let Q be a quiver. Suppose that γ_1, γ_2 are vertices in Q . Suppose that a is an arrow in Q , whose source is γ_1 , and tail is γ_2 , and that a is the only such arrow. Let r_1, r_2 be natural numbers, whose sum is r . There is a graded module $C_{r_1, r_2} = C_{r_1, r_2}(a)$ for $\mathcal{D}_Q(n, r)$, whose graded pieces are the $\mathcal{S}_{V(Q)}(n, r)$ -modules,*

$$\begin{aligned} & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_2}^{\otimes r_2 + 1} + \Delta_{\gamma_1}(r_1, 1) \otimes E_{\gamma_2}^{\otimes r_2 - 1} \\ & \Delta_{\gamma_1}(r_1 - 1, 1) \otimes E_{\gamma_2}^{\otimes r_2} + \Delta_{\gamma_1}(r_1, 1^2) \otimes E_{\gamma_2}^{\otimes r_2 - 2} \\ & \dots \\ & \Delta_{\gamma_1}(r_1 - 1, 1^{r_2 - 1}) \otimes E_{\gamma_2}^{\otimes 2} + \Delta_{\gamma_1}(r_1, 1^{r_2}) \\ & \Delta_{\gamma_1}(r_1 - 1, 1^{r_2}) \otimes E_{\gamma_2}, \end{aligned}$$

in degrees $0, 1, \dots, r_2 + 1$.

There is a natural homomorphism $d_{r_1, r_2} = d_{r_1, r_2}(a)$ of degree one, from C_{r_1-1, r_2+1} to C_{r_1, r_2} . The sequence of maps $\{d_{r_1, r_2} | r_1 + r_2 = r\}$ defines a chain complex, $\mathcal{P}_r(a)$, given by,

$$\dots \rightarrow 0 \rightarrow C_{1, r-1} \rightarrow C_{2, r-2} \rightarrow \dots \rightarrow C_{r-1, 1} \rightarrow C_{r, 0} \rightarrow 0 \rightarrow \dots$$

The homology of this complex at term C_{r_1, r_2} is isomorphic as a $\mathcal{D}_Q(n, r)$ -module, to the standard module $\Delta_{\gamma_1}(r_1, 1^{r_2})$ for $\mathcal{S}_{\gamma_1}(n, r)$, concentrated in degree $r_1 - 1$.

Proof:

Consider the $\mathcal{D}_Q(n, r) - \mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)$ -bimodule,

$$X_{r_1, r_2} = \mathcal{S}_Q(n, r) \xi_{r_1, r_2} \oplus (\mathcal{S}_Q(n, r-1) \xi_{r_1-1, r_2} \otimes \bigwedge (n, 1)_{\gamma_2, \gamma_1}),$$

given as a quotient of the $\mathcal{D}_Q(n, r) - \mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)$ -bimodule,

$$\mathcal{D}_Q(n, r) \xi_{r_1, r_2},$$

modulo terms of higher degree. Note that

$$Y_{r_1, r_2} = (\mathcal{S}_Q(n, r-1) \xi_{r_1-1, r_2} \otimes \bigwedge_{\gamma_2, \gamma_1} (n, 1)),$$

is a sub-bimodule of X_{r_1, r_2} . Let us define the $\mathcal{D}_Q(n, r)$ -module,

$$U_{r_1, r_2} = X_{r_1, r_2} \bigotimes_{\mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)} (\Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2}),$$

which contains the submodule,

$$V_{r_1, r_2} = Y_{r_1, r_2} \bigotimes_{\mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)} (\Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2}).$$

Thus, U_{r_1, r_2} is a graded module, whose graded pieces are the $\mathcal{S}_{V(Q)}(n, r)$ -modules,

$$\begin{aligned} & \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_2}^{\otimes r_2+1} + \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_1} \otimes E_{\gamma_2}^{\otimes r_2-1} \\ & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_1} \otimes E_{\gamma_2}^{\otimes r_2} + \Delta_{\gamma_1}(r_1) \otimes E_{\gamma_1}^{\otimes 2} \otimes E_{\gamma_2}^{\otimes r_2-2} \\ & \dots \\ & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_1}^{r_2-1} \otimes E_{\gamma_2}^{\otimes 2} + \Delta_{\gamma_1}(r_1, 1^{r_2}) \\ & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_1}^{r_2} \otimes E_{\gamma_2}, \end{aligned}$$

in degrees $0, 1, \dots, r_2 + 1$. Let us define,

$$O_{r_1, r_2} = \mathcal{D}_Q(n, r) i_{r_1-1, r_2+1} V_{r_1, r_2},$$

a submodule of U_{r_1, r_2} . As in lemma 167, $D_{r_1, r_2} = V_{r_1, r_2}/O_{r_1, r_2}$ is a graded module, whose graded pieces are the $\mathcal{S}_{V(Q)}(n, r)$ -modules,

$$\begin{aligned} & \Delta_{\gamma_1}(r_1 - 1) \otimes E_{\gamma_2}^{\otimes r_2+1} \\ & \Delta_{\gamma_1}(r_1 - 1, 1) \otimes E_{\gamma_2}^{\otimes r_2} \\ & \dots \\ & \Delta_{\gamma_1}(r_1 - 1, 1^{r_2-2}) \otimes \otimes E_{\gamma_2}^{\otimes 2} \\ & \Delta_{\gamma_1}(r_1 - 1, 1^{r_2}) \otimes E_{\gamma_2}, \end{aligned}$$

in degrees $0, 1, \dots, r_2$. Let us define,

$$P_{r_1, r_2} = \mathcal{D}_Q(n, r) i_{r_1, r_2} (U_{r_1, r_2}/O_{r_1, r_2}),$$

a submodule of $M_{r_1, r_2}/O_{r_1, r_2}$. We now define,

$$C_{r_1, r_2} = U_{r_1, r_2}/(O_{r_1, r_2} + P_{r_1, r_2}),$$

which contains D_{r_1, r_2} as a submodule. The graded pieces of C_{r_1, r_2} are visibly those described in the statement of the theorem.

Let η be the element, $1_{\mathcal{S}_{\gamma_1}(n, r_1-1)} \otimes 1_{M_n(k)} \otimes 1_{\mathcal{S}_{\gamma_2}(n, r_2+1)}$ of the $\mathcal{S}_{\gamma_1}(n, r_1 - 1) \otimes \mathcal{S}_{\gamma_2}(n, r_2 + 1)$ - $\mathcal{S}_{\gamma_1}(n, r_1) \otimes \mathcal{S}_{\gamma_2}(n, r_2)$ - bimodule,

$$\mathcal{S}_{\gamma_1}(n, r_1 - 1) \otimes \bigwedge_{\gamma_2, \gamma_1} (n, 1) \otimes \mathcal{S}_{\gamma_2}(n, r_2).$$

Then multiplication by η defines a map from X_{r_1-1, r_2+1} to Y_{r_1, r_2} . By restriction, there is a map,

$$d_{r_1, r_2} : C_{r_1-1, r_2+1} \rightarrow C_{r_1, r_2}.$$

The kernel of d_{r_1+1, r_2-1} is equal to the submodule $D_{r_1-1, r_2+1} + \Delta_{\gamma_1}(r_1, 1^{r_2})$ of C_{r_1, r_2} , and the image of d_{r_1, r_2} is isomorphic to D_{r_1, r_2} . The chain complex $\mathcal{P}_r(a)$ defined by the sequence of maps, thus has homology $\Delta_{\gamma_1}(r_1, 1^{r_2})$, at term C_{r_1, r_2} . \square

Theorem 169 *The Grothendieck character of the complex $\mathcal{P}_r(a)$ of $\mathcal{D}_Q(n, r)$ -modules describes the power sum p_r , at γ_1 .*

Proof:

The Grothendieck character of a standard module $\Delta(\lambda)$ is given by the Schur function s_λ . There is a formula for the power sum p_r , given by ([54], I.4, example 10),

$$p_r = s_{(r)} - s_{(r-1,1)} + s_{(r-2,1^2)} - \dots \pm s_{(1^r)}.$$

The Grothendieck character of $\mathcal{P}_r(a)$ thus describes the power sum p_r . \square

I. Frenkel, N. Jing and W. Wang have given a description of the homogeneous vertex operator construction of the basic representation of an affine Lie algebra of type ADE , via wreath products of finite group algebras [33]. The characters of symmetric groups which are used in that paper correspond in symmetric function theory to elementary symmetric functions, to complete symmetric functions, and to power sums.

We have observed here that there are objects in the derived category of a Schiver double which correspond to all these functions.

In fact, upon studying Frenkel, Jing and Wang's construction more carefully, one realises that the Schiver double afforded to a simply-laced Dynkin diagram Γ , underlies a category for a vertex representation for the affinization of the Kac-Moody Lie algebra defined by Γ , at least when Γ is ordinary/affine, of type ADE .

By this, we only mean that we can describe a category whose complexified Grothendieck group is the vertex representation, and we can describe functors which correspond to the vertex operators, upon passing to the Grothendieck group. The category is a direct product of derived categories for Schiver doubles, and the functors are described by certain complexes of bimodules. Note that we have *not* explored the extent to which relations in the affine Lie algebra lift to relations between functors.

Conjecture 165 is comfortably consistent with this categorical perspective. The blocks of Hecke algebras are already well known to be categories which describe the basic representation for $\widehat{\mathfrak{sl}}_p$, as has most elegantly been described by I. Grojnowski [40], following theory of A. Lascoux, B. Leclerc, and J-Y. Thibon [52], as well as S. Ariki [3], and A. Kleshchev [47], [48]. In Grojnowski's article, relations in the affine Lie algebra do lift to relations between functors.

In general, we expect an equivalence of $\widehat{\mathfrak{sl}}_p$ -categorifications,

$$D^b \left(\bigoplus_{w \geq 0, s \in \widehat{W}/W} \mathcal{D}_{A_{p-1}}(w, w) \right) \cong D^b \left(\bigoplus_{r \geq 0} \mathcal{H}_q(\Sigma_r) \right),$$

although it is unclear to me what this means precisely; so far, only \mathfrak{sl}_2 -categorifications possess an axiomatic definition and a general theory [12]. Above, W is the Weyl group of \mathfrak{sl}_p , and \widehat{W} is the corresponding affine Weyl group.

Chapter X

Schiver doubles of type A_∞ .

We consider the infinite Dynkin quiver A_∞ :



We prove that the module category of $\mathcal{D}_{A_\infty}(w, w)$ is a highest weight category, and then conjecture that Rock blocks $k\mathbf{B}_{\rho, w}^{S_q}$, of q -Schur algebras, are Morita equivalent to certain subquotients of the algebra $\mathcal{D}_{A_\infty}(w, w)$, defined in chapter 8 (conjecture 178).

We also describe a walk along $\mathcal{D}_{A_\infty}(n)$, analogous to J.A. Green's walk along the Brauer tree (theorem 182).

Schiver bialgebras for A_∞ .

Consider the infinite Dynkin quiver A_∞ , which has vertex set

$$V_\infty = \{v_i, i \in \mathbb{Z}\},$$

and edge set $E_\infty = \{e_i, i \in \mathbb{Z}\}$. The source of any edge e_i is the vertex v_{i+1} , and its tail is v_i .

We present the Schiver bialgebra corresponding to A_∞ . This bialgebra amplifies the category of chain complexes over k , as the classical Schur bialgebra amplifies the category of vector spaces over k .

Definition 170 Let $\underline{\Lambda}(n, r)$ (respectively $\underline{\Lambda}'(n, r)$) be the set $\{\underline{\lambda} = (\lambda_i)_{i \in \mathbb{Z}}\}$ of sequences of partitions, with n parts or fewer, whose sizes sum to r .

For two elements $\underline{\lambda}, \underline{\mu}$ of $\underline{\Lambda}(n, r)$ (respectively $\underline{\Lambda}'(n, r)$), let $\underline{\lambda} \leq \underline{\mu}$ if and only if, the sequence $(\lambda_i)_{i \in \mathbb{Z}}$ can be obtained from the sequence $(\mu_i)_{i \in \mathbb{Z}}$ in finitely many steps, by repeatedly either,

- (1) removing a box from the Young diagram of μ_i , and replacing it lower down on the same Young diagram, to create a new partition, or
- (2) removing a box from the Young diagram of μ_i , and adding it on to the Young diagram of μ_{i-1} (respectively μ_{i+1}), to create a new partition.

The posets $\underline{\Lambda}(n, r), \underline{\Lambda}'(n, r)$ generalise the poset of partitions of r with n parts or fewer, with the dominance ordering.

Recall that the the degree zero part of the algebra $\mathcal{S}_{A_\infty}(n, r)$ is equal to,

$$\mathcal{S}_{V_\infty}(n, r) \cong \bigoplus_{(r_i)_{i \in \mathbb{Z}}, \sum r_i = r} \left(\bigotimes_{i \in \mathbb{Z}} \mathcal{S}(n, r_i) \right).$$

Recall further, that the module category $\mathcal{S}(n, r) - \text{mod}$ for the classical Schur algebra, is a highest weight category, whose standard modules $\Delta(\lambda)$ are named Weyl modules, indexed by partitions of r with n parts or fewer. We write $\nabla(\lambda)$ for the costandard $\mathcal{S}(n, r)$ -module corresponding to a partition λ of r .

Definition 171 Let $\underline{\lambda} = (\lambda_i)_{i \in \mathbb{Z}}$ be a sequence of partitions, finitely many of which are non-empty, whose sizes are given by the sequence $(r_i)_{i \in \mathbb{Z}}$. Let $r = \sum_{i \in \mathbb{Z}} r_i$.

The standard module, $\Delta_1(\underline{\lambda})$ for $\mathcal{S}_{A_\infty}(n, r)$, is given by,

$$\mathcal{S}_{A_\infty}(n, r) \bigotimes_{\mathcal{S}_{V_\infty}(n, r)} \left(\bigotimes_{i \in \mathbb{Z}} \Delta(\lambda_i) \right).$$

The costandard module, $\nabla_1(\underline{\lambda})$ for $\mathcal{S}_{A_\infty}(n, r)$, is the $\mathcal{S}_{V_\infty}(n, r)$ -module,

$$\bigotimes_{i \in \mathbb{Z}} \nabla(\lambda_i).$$

The costandard module, $\nabla_2(\underline{\lambda})$ for $\mathcal{S}_{A_\infty}(n, r)$, is given by,

$$\text{Hom}_{\mathcal{S}_{V_\infty}(n, r)} \left(\mathcal{S}_{A_\infty}(n, r), \bigotimes_{i \in \mathbb{Z}} \nabla(\lambda_i) \right).$$

The standard module, $\Delta_2(\underline{\lambda})$ for $\mathcal{S}_{A_\infty}(n, r)$, is the $\mathcal{S}_{V_\infty}(n, r)$ -module,

$$\bigotimes_{i \in \mathbb{Z}} \Delta_2(\lambda_i).$$

Theorem 172 Let $n \geq r$ be natural numbers.

The module category $\mathcal{S}_{A_\infty}(n, r) - \text{mod}$, is a highest weight category with respect to the poset $\underline{\Lambda}(n, r)$. Given an element $\underline{\lambda}$ of $\underline{\Lambda}(n, r)$, the corresponding standard module is $\Delta_1(\underline{\lambda})$, and the corresponding costandard module is $\nabla_1(\underline{\lambda})$.

The module category $\mathcal{S}_{A_\infty}(n, r) - \text{mod}$, is also a highest weight category with respect to the poset $\underline{\Lambda}'(n, r)$. Given an element $\underline{\lambda}$ of $\underline{\Lambda}'(n, r)$, the corresponding standard module is $\Delta_2(\underline{\lambda})$, and the corresponding costandard module is $\nabla_2(\underline{\lambda})$.

Proof:

We describe only the quasi-hereditary structure with respect to $\underline{\Lambda}(n, r)$. The quasi-hereditary structure with respect to $\underline{\Lambda}'(n, r)$ may be understood similarly.

Throughout this proof, we indiscriminately use the Ringel self-duality of $\mathcal{S}(n, t)$, for $n \geq t$ (theorem 17).

We first show that the projective cover of a simple module $L(\underline{\lambda})$ is filtered by standard modules $\Delta_1(\underline{\mu})$, with $\underline{\lambda} \trianglelefteq \underline{\mu}$. Note that if $P(\underline{\lambda})$ is the projective $\mathcal{S}_{V_\infty}(n, r)$ module, with top $L(\underline{\lambda})$, then $\mathcal{S}_{A_\infty}(n, r) \otimes_{\mathcal{S}_{V_\infty}(n, r)} P(\underline{\lambda})$ is the projective cover of $L(\underline{\lambda})$ as a $\mathcal{S}_{A_\infty}(n, r)$ -module. Since $P(\underline{\lambda})$ is filtered by $\Delta(\underline{\mu})$, with $\underline{\lambda} \trianglelefteq \underline{\mu}$, and for $n \geq t$, the functor $\otimes_{\mathcal{S}_{V_\infty}(n, t)} \bigvee(n, t)$ is exact on the category of Δ -filtered $\mathcal{S}_{V_\infty}(n)$ -modules, we deduce that $\mathcal{S}_{A_\infty}(n) \otimes_{\mathcal{S}_{V_\infty}(n)} P(\underline{\lambda})$ is filtered by $\Delta_1(\underline{\mu})$, with $\underline{\lambda} \trianglelefteq \underline{\mu}$.

We secondly remark that the simple composition factors of $\Delta_1(\underline{\lambda})$ are indexed by elements $\underline{\mu}$ of $\underline{\Delta}(n, r)$, such that $\underline{\mu} \trianglelefteq \underline{\lambda}$. This is a consequence of the branching rule for classical Schur algebras, as well as the quasi-heredity of $\mathcal{S}_{V_\infty}(n)$.

Thirdly, that the costandard modules relevant to this highest weight structure are the $\nabla_1(\underline{\lambda})$'s is now visible. By duality, we need only observe that $\Delta_2(\underline{\lambda})$ is the largest quotient of the projective cover of $L(\underline{\lambda})$ for whose composition factors $L(\underline{\mu})$ (excepting the top $L(\underline{\lambda})$), the multipartition $\underline{\mu}$ is strictly smaller than $\underline{\lambda}$, with respect to the ordering on $\underline{\Delta}'(n, r)$. This is apparent from the structure we have already described on the projective cover of $L(\underline{\lambda})$. \square

We consider the Schiver double, $\mathcal{D}_{A_\infty}(n)$. An immediate corollary of theorem 157 is,

Theorem 173 *The action of the infinite dihedral group D_∞ as graph automorphisms of A_∞ lifts to an action of D_∞ as algebra automorphisms on $\mathcal{D}_{A_\infty}(n)$. \square*

Let

$$\begin{aligned} '[1] : \underline{\Delta}(n, r) &\rightarrow \underline{\Delta}(n, r), \\ (\underline{\lambda}'[1])_i &= \lambda'_{i-1}, \end{aligned}$$

be the map on $\underline{\Delta}(n, r)$, which shifts a sequence by 1, and then conjugates each entry in the sequence.

Lemma 174 *Let $n \geq r$ be natural numbers.*

There is an isomorphism, $\Delta_1(\underline{\lambda}) \cong \nabla_2(\underline{\lambda}'[1])$.

Proof:

The natural sequence of homomorphisms,

$$\Delta_1(\underline{\lambda})^{op} \otimes \Delta_1(\underline{\lambda}'[1]) \rightarrow \Delta_1(\underline{\lambda})^{op} \otimes_{\mathcal{S}_{A_\infty}(n, r)} \Delta_1(\underline{\lambda}'[1])$$

$$\begin{aligned}
&\cong \left(\bigotimes_{i \in \mathbb{Z}} \Delta(\lambda_i)^{op} \right) \otimes_{\mathcal{S}_{V_\infty}(n,r)} \left(\bigotimes_{i \in \mathbb{Z}} V(n, r_i) \right) \otimes_{\mathcal{S}_{V_\infty}(n,r)} \left(\bigotimes_{i \in \mathbb{Z}} \nabla(\lambda'_i) \right) \\
&\cong \left(\bigotimes_{i \in \mathbb{Z}} \Delta(\lambda_i)^{op} \right) \otimes_{\mathcal{S}_{V_\infty}(n,r)} \left(\bigotimes_{i \in \mathbb{Z}} \Delta(\lambda_i) \right) \cong k,
\end{aligned}$$

defines a non-degenerate bilinear form,

$$\langle, \rangle: \Delta_1(\underline{\lambda})^{op} \times \Delta_1(\underline{\lambda}'[1]) \rightarrow k,$$

such that $\langle x \circ s, y \rangle = \langle x, s \circ y \rangle$, for $s \in \mathcal{S}_{A_\infty}(n, r)$.

Since the dual of the $\mathcal{S}_{A_\infty}(n, r)$ -module $\Delta_1(\underline{\lambda})$, is isomorphic to $\nabla_2(\underline{\lambda})$, the existence of such a bilinear form seals the proof of the lemma. \square

Theorem 175 *Let $n \geq r$ be natural numbers.*

The Schiver algebra $\mathcal{S}_{A_\infty}(n, r)$ is Ringel self-dual. Indeed, Ringel duality exchanges the two highest weight structures we have introduced on $\mathcal{S}_{A_\infty}(n, r) - \text{mod}$.

The module category $\mathcal{D}_{A_\infty}(n, r) - \text{mod}$ is a highest weight category, with respect to the poset $\underline{\Lambda}(n, r)$. Given an element $\underline{\lambda}$ of $\underline{\Lambda}(n, r)$, the corresponding standard module is $\Delta_1(\underline{\lambda})$.

Furthermore, $\mathcal{D}_{A_\infty}(n, r) - \text{mod}$ is a highest weight category, with respect to the poset $\underline{\Lambda}'(n, r)$. Given an element $\underline{\lambda}$ of $\underline{\Lambda}'(n, r)$, the corresponding costandard module is $\nabla_2(\underline{\lambda})$.

Indeed, $\mathcal{D}_{A_\infty}(n, r)$ is Ringel self-dual, and Ringel duality exchanges these two highest weight structures on $\mathcal{D}_{A_\infty}(n, r) - \text{mod}$.

Proof:

As a consequence of lemma 174, and theorem 172, the regular representation of $\mathcal{S}_{A_\infty}(n, r)$ can be filtered by ∇_2 's, as well as filtered by Δ_2 's.

Thus, the regular representation is a full tilting module for $\mathcal{S}_{A_\infty}(n, r)$ with respect to $\underline{\Lambda}'(n, r)$, and indeed $\mathcal{S}_{A_\infty}(n, r)$ is Ringel self-dual. Ringel duality exchanges the two highest weight structures we have defined on $\mathcal{S}_{A_\infty}(n, r)$, because the functor $\text{Hom}_{\mathcal{S}_{A_\infty}(n, r)}(\mathcal{S}_{A_\infty}(n, r), -)$ must affect co-standard modules to become standard modules. \square

Remark 176 Let \mathcal{C} be a highest weight category, with poset Λ , and let $\Pi = \Gamma \cap \Omega$ be the intersection of an ideal $\Gamma \subset \Lambda$ and a coideal $\Omega \in \Lambda$. Then there is a canonically defined highest weight category $\mathcal{C}(\Pi)$, whose poset is Π (see theorem 3). So long as Π is a finite set, \mathcal{C} is the module category of a quasi-hereditary algebra.

Let p be a natural number. We define $\Gamma_p(n, r)$ (respectively $\Gamma'_p(n, r)$) to be the ideal (respectively coideal), of sequences $(\lambda_i) \in \underline{\Lambda}(n, r)$ (respectively $\underline{\Lambda}'(n, r)$), all of whose entries λ_i are zero, for $i > p - 1$.

Let us define $\Omega_p(n, r)$ (respectively $\Omega'_p(n, r)$), to be the coideal (respectively ideal), of sequences $(\lambda_i) \in \underline{\Lambda}(n, r)$ (respectively $\underline{\Lambda}'(n, r)$), all of whose entries λ_i are zero, for $i < 0$.

Let $\Pi_p(n, r) = \Gamma_p(n, r) \cap \Omega_p(n, r)$ (respectively $\Pi'_p(n, r) = \Gamma'_p(n, r) \cap \Omega'_p(n, r)$), be the set of sequences $(\lambda_i) \in \underline{\Lambda}(n, r)$ (respectively $\underline{\Lambda}'(n, r)$), all of whose entries λ_i are zero, for $i < 0$, and $i > p - 1$.

For $n \geq r$, let $\mathcal{Q}_p(n, r)$ (respectively $\mathcal{Q}'_p(n, r)$), be the quasi-hereditary subquotient of $\mathcal{D}_{A_\infty}(n, r)$, whose poset is $\Pi_p(n, r)$ (respectively $\Pi'_p(n, r)$), and whose module category is the highest weight category, $(\mathcal{D}_{A_\infty}(n, r) - \text{mod})(\Pi_p(n, r))$ (respectively $(\mathcal{D}_{A_\infty}(n, r) - \text{mod})(\Pi'_p(n, r))$).

Remark 177 By theorem 175, Ringel duality exchanges the quasi-hereditary algebras, $\mathcal{Q}_p(n, r)$ and $\mathcal{Q}'_p(n, r)$.

The quiver A_∞ possesses an orientation-reversing automorphism, which exchanges vertex v_0 , and vertex v_{p-1} . By theorem 173, this automorphism lifts to an automorphism Θ of $\mathcal{D}_{A_\infty}(n, r)$.

The automorphism Θ provides an isomorphism between $\mathcal{Q}_p(n, w)$ and $\mathcal{Q}'_p(n, w)$. Therefore, $\mathcal{Q}_p(n, w)$ is Ringel self-dual.

We may now formulate a generalisation of conjecture 165 to q -Schur algebras.

Let $q \in k^\times$, and let p be the least natural number such that,

$$1 + q + \dots + q^{p-1} = 0.$$

Let n, w be natural numbers, such that $n \geq w$.

Conjecture 178 *The quasi-hereditary algebra $\mathcal{Q}_p(n, w)$ is Morita equivalent to any Rock block $k\mathbf{B}_{\rho, w}^{S_q}$ of a q -Schur algebra, whose weight is w .*

Indeed, $\mathcal{Q}_p(n, w)$ is derived equivalent to any block $k\mathbf{B}_{\tau, w}^{S_q}$ of a q -Schur algebra, whose weight is w .

How far does this conjecture generalise ?

Question 179 *Can all blocks of q -Schur algebras of weight w be \mathbb{Z}_+ -graded, so that the degree zero part is Morita equivalent to the James adjustment algebra ?*

Remark 180 Conjecture 165 and conjecture 178 are related as follows:

For $n \geq w$, the double $\mathcal{D}_{A_{p-1}}(n, w)$ is obtained from $\mathcal{Q}_p(n, w)$ by cutting at the idempotent $j_p \in \mathcal{Q}_p(n, r)$ corresponding to the subset $\Pi_{p-1}(n, w) \subset \Pi_p(n, w)$.

We should thus define the Specht modules for $\mathcal{D}_{A_{p-1}}(n, w)$, to be those modules $j_p \cdot \Delta$, where Δ is a standard module for $\mathcal{Q}_p(n, w)$.

It is now possible to deduce the following result from theorem 132, the definition of standard modules for $\mathcal{D}_{A_\infty}(w, w)$, and formula 59, for the decomposition matrix of $K\mathbf{B}_{\rho, w}^{\mathcal{H}_q}$, where q is a p^{th} root of unity.

Corollary 181 *Let k be a field of characteristic p . Then the symmetric group Rock block $k\mathbf{B}_{\rho, w}^\Sigma$ has the same decomposition matrix as $\mathcal{D}_{A_{p-1}}(w, w)$.*
□

One proof uses the Littlewood-Richardson rule, concerning tensor products of modules for the Schur algebra. Conjecture 165 thus structurally clarifies formula 59 of Chuang-Tan, and Leclerc-Miyachi.

Walking along A_∞ .

The super-algebra P_{A_∞} , is endowed with a natural differential d of degree 1, given by the infinite sum, $\sum_{i \in \mathbb{Z}} e_i$, of all edges. Indeed, the complex,

$$\dots\dots\dots \rightarrow P_{A_\infty} \rightarrow P_{A_\infty} \rightarrow P_{A_\infty} \rightarrow \dots\dots\dots,$$

with differential given by right multiplication by d , is a linear exact sequence of left P_{A_∞} -modules.

In the last passage of this letter, we lift this elegant differential structure on P_{A_∞} , to the super-bialgebra $\mathcal{S}_{A_\infty}(n)$, and its double $\mathcal{D}_{A_\infty}(n)$. We call the resulting chain complex a “walk along A_∞ ”, since it generalises a homological structure discovered by J.A. Green on blocks of finite groups of cyclic defect: the “walk around the Brauer tree”.

Let d be the differential on $P_{A_\infty}(n)$ of degree 1, given by

$$d = (0^{\times V}) \times (1^{\times E}) \in (End_k(k^n))^{\times V} \times (End_k(k^n))^{\times E}.$$

Let d_r be the differential on $P_{A_\infty}(n)^{\otimes r}$ of degree 1, given by

$$d \otimes 1 \otimes \dots \otimes 1 + 1 \otimes d \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes d.$$

Note that d_r is invariant under the action of the symmetric group Σ_r , and so d_r is a differential on the Schiver super-bialgebra $\mathcal{S}_{A_\infty}(n, r)$.

Theorem 182 (“Walk along A_∞ ”) *The chain complex,*

$$\dots\dots\dots \rightarrow \mathcal{S}_{A_\infty}(n, r) \rightarrow \mathcal{S}_{A_\infty}(n, r) \rightarrow \mathcal{S}_{A_\infty}(n, r) \rightarrow \dots\dots\dots,$$

with differential given by right multiplication by d_r , is a linear exact sequence of left $\mathcal{S}_{A_\infty}(n, r)$ -modules. The chain complex,

$$\dots\dots\dots \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \mathcal{D}_{A_\infty}(n, r) \rightarrow \dots\dots\dots,$$

with differential given by right multiplication by d_r , is a linear exact sequence of left $\mathcal{D}_{A_\infty}(n, r)$ -modules.

Proof:

The classical Koszul complex on $\text{End}(k^n)^*$ is the acyclic chain complex,

$$\mathcal{A}(n) \otimes \bigwedge(n),$$

whose differential is given by,

$$d : \mathcal{A}(n, l) \otimes \bigwedge(n, m) \rightarrow \mathcal{A}(n, l+1) \otimes \bigwedge(n, m-1),$$

$$y_1 \dots y_l \otimes x_1 \wedge \dots \wedge x_m \mapsto$$

$$\sum_{i=1}^m (-1)^{i-1} y_1 \dots y_l x_i \otimes x_1 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_m.$$

Its dual is an acyclic chain complex,

$$\mathcal{S}(n) \otimes \bigvee(n).$$

Tensoring together \mathbb{Z} copies of this dual Koszul complex, and forming the total complex, in degree r we obtain an acyclic chain complex which corresponds precisely to the first exact sequence of theorem 182.

Tensoring together \mathbb{Z} copies of the Koszul complex, along with \mathbb{Z} copies of the dual Koszul complex, in degree r we obtain an exact sequence which corresponds precisely to the second exact sequence of theorem 182. \square

Remark 183 The super-algebra, P_{A_∞} , is a Koszul algebra [4]. Its Koszul dual is the path algebra kA_∞^{op} on the quiver A_∞ , with opposite orientation.

The Schiver super-algebra $\mathcal{S}_{A_\infty}(n)$ is not a Koszul algebra (unless k is a field of characteristic zero). Its degree zero part,

$$\mathcal{S}_{V_\infty}(n) = \bigotimes_{v \in \mathbb{Z}} \mathcal{S}(n),$$

is not semisimple. However, it does possess a linear resolution, and the algebra,

$$Ext_{\mathcal{S}_{A_\infty}(n)}^*(\mathcal{S}_{V_\infty}(n), \mathcal{S}_{V_\infty}(n)),$$

is isomorphic to $\mathcal{S}(kA_\infty^{op})(n)$, while the path algebra kA_∞^{op} is concentrated in parity zero.

The algebra,

$$Ext_{\mathcal{D}_{A_\infty}(n)}^*(\mathcal{S}_{V_\infty}(n), \mathcal{S}_{V_\infty}(n)),$$

is isomorphic to the algebra $\mathcal{S}(\Pi_{A_\infty})(n)$, where Π_{A_∞} is the preprojective algebra on the graph A_∞ , concentrated in parity zero.

A similar statement is true, relating $\mathcal{D}_{\tilde{A}_{p-1}}(n)$ and $\mathcal{S}(\Pi_{\tilde{A}_{p-1}})(n)$. I prove this, along with various stronger results, in my paper “On seven families of algebras” [69]. Sending p to infinity, one obtains theorems for the algebras associated to A_∞ .

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